THE
MATHEMATICAL GAZETTE

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THE MATHEMATICAL ASSOCIATION.

THE Annual General Meeting of the Mathematical Association was held at the Institute of Education, Malet Street, on 1st, 2nd, and 3rd January, 1948. During the course of the meeting, the pleasure of the Association at the renewal of the pre-war link with the Institute was expressed by Professor E. H. Neville and Mr. Hope-Jones.

On 1st January, the business meeting was held at 2 p.m., the President, Professor G. B. Jeffery, Director of the Institute of Education, in the chair. The Report of the Council for 1947 was adopted.* The election of Sir Harold Spencer Jones, Sc.D., F.R.S., as President for 1948 was announced. The following were elected as Vice-presidents: Dr. W. L. Ferrar, Professor W. V. D. Hodge, Mr. W. Hope-Jones, Professor G. B. Jeffery, Professor E. H. Neville, Mr. A. W. Siddons, Mr. K. S. Snell, Mr. C. O. Tuckey, Professor G. N. Watson, Sir Edmund Whittaker. Mr. J. B. Morgan was elected as Treasurer, and the meeting endorsed a resolution passed by the Council thanking Mr. K. S. Snell for his 12 years' service as Treasurer. The Secretaries, the Librarian, the Editor of the *Mathematical Gazette*, and the Auditor were re-elected. The following were elected to serve on the Council: Professor T. A. Brown, Mr. C. T. Daltry, Mr. F. W. Kellaway, Dr. F. G. Maunsell, Dr. E. A. Maxwell, Mr. M. A. Porter, Mr. A. Robson, Dr. J. Topping, Mrs. E. M. Williams.

At 2.45 p.m. the President delivered his Presidential Address, " Mathematics as an Educational Experience " †; at 5 p.m. Professor E. H. Neville read a paper on " What is a Number? ".

On 2nd January, at 10 a.m., Mr. J. T. Combridge opened a discussion on " Replacing Mathematics by Reckoning ", and at 11 a.m. Professor H. W. Turnbull read a paper on " Intuitive Geometry ". At 2 p.m. a discussion was opened on " The Typography of Mathematical Textbooks " and at 5 p.m. Professor A. C. Aitken gave " A Demonstration of Arithmetic ".

On 3rd January, at 10 a.m., a discussion on " Mathematics in School Examinations, 1950 " was opened by Mr. W. J. Langford, Mr. M. W. Brown, Miss F. M. A. Pendry and Dr. E. A. Maxwell.

A Publishers' Exhibition and an Exhibition of Visual Aids material connected with certain selected topics, were open during the three days.

* See pp. 2–5. † See pp. 6–14.

REPORT OF THE COUNCIL FOR THE YEAR 1947.

Membership.

During the period from 1st January to 31st October, 1947, 110 new members have been admitted to the Association, of whom 26 are junior members. The total membership at the end of October stood at 2256, of whom 5 are Honorary Members and 176 are Life Members.

It is with deep regret that the Council reports the death of two of the Association's most distinguished members, Professor G. H. Hardy and Professor A. N. Whitehead. Each had served the Association as President, and their Presidential Addresses reveal the strong interest taken by these two great mathematicians in problems of teaching. In addition, Professor Hardy's numerous contributions to the *Gazette*, particularly between 1900 and 1914, were of outstanding value in helping to reform the teaching of mathematical analysis in this country.

The Council also reports with regret the death of the following members: Mr. C. T. L. Caton (1935), Dr. S. Weikersheimer (1946), Mr. F. F. Ellis (1926). Mr. C. T. Lear Caton was a joint-secretary of the Midland Branch, where he will be greatly missed, and was also a member of the Teaching Committee.

The Council reports that, owing to the excellent work of Mr. M. A. Porter, who undertook the duties of Membership Secretary at the beginning of the year, the list of members, which contained many anomalies and inaccuracies due to the changing conditions of the war years, has been brought up to date. A new printed list of members is in process of going through the press. The Council requests all members to co-operate by sending to Mr. Porter, or to one of the Secretaries, prompt information of any alterations in addresses, employment, etc., which should be recorded.

Finance.

Receipts and expenditure have again both advanced very considerably. The increase of £600 in receipts is made up by about £200 in increased subscriptions due to larger membership, and most of the remainder from the much greater sale of *Gazettes* and Reports. The sale from the latter is particularly large, because some reports were out of print a year ago and orders accumulated. Expenditure increased by over £350, largely on account of increased costs of printing and of clerical assistance. The Teaching Committee is very active, and hence the expenses of its sub-committees remain large, but this a healthy sign. There is a deficit of £247 on the year. This is being met out of funds accumulated during the war in expectation of renewed activity. If we are to maintain expenditure at the present level, it is essential that the rate of increase of membership should be as large during the coming year as during the last two years.

The Branches.

The Branches Committee held a meeting in April, and useful information was exchanged on the activities of the Branches. A further meeting was held at the end of November.

Most Branches record interesting and well-attended meetings, in spite of difficulties occasioned by the weather and the fuel crisis.

A feature of the present activities of the Branches is a substantial increase of members, especially among younger members and men and women from the Emergency Training Colleges.

North Wales and the Swansea Branch have experienced considerable

difficulties, but the Council is glad to welcome the newly-formed Leicester and County Branch.

Considerable thought is being devoted in the Branches to the Alternative Mathematics syllabuses of the various examining bodies, while the inter-relations between the various types of secondary education provide further useful subjects for discussion.

London records even larger attendances, and Mr. Durell's presidential address on " The Transition from School to University Mathematics " and Mr. Sawyer's talk on " Mathematics for the Secondary Modern School " were both much appreciated.

Southampton has had a talk by Professor Cave-Browne-Cave on " The Mechanics of the Fun-Fair ", and has arranged a full programme for the session. Cardiff has also reported a successful season.

Manchester has received a report of the Sub-Committee on the possibility of Standardisation of the Fundamental Processes of Arithmetic.

Liverpool has had an address by Professor Newman on " Mathematics and Machines ", while Northern Ireland has discussed the question of mathematical curricula.

The overseas Branches have also continued their active work. New South Wales has had lectures on the history of mathematics and on Standardised Tests. The *Australian Mathematics Teacher* has completed a successful two years of existence and has received support throughout Australia. Queensland and Victoria are still flourishing, and the new Branch at Auckland, N.Z., has had successful meetings.

The Mathematical Gazette.

Vol. XXXI is of the size which we must now regard as normal ; production difficulties still exist, and delays have not yet been abolished.

Vol. XXXII should see the publication of No. 300 (July 1948). It is intended to make this a special number. Items of particular interest from earlier *Gazettes* will be reprinted, and promises of new contributions written for the purpose have been obtained from a number of distinguished mathematicians, including Mr. C. V. Durell, Professor J. E. Littlewood, Professor S. Mandelbrojt, Professor E. H. Neville, Mr. A. W. Siddons and Sir D'Arcy Wentworth Thompson.

The Library.

The Library has remained throughout the year comparatively inaccessible at Douai Abbey, and service has been uncertain, but an agreement has been concluded which is designed not only to bring the present difficulties to an end early in 1948, but to give the Association in due course a permanent home for its books : accommodation and routine service are to be provided by the University of Reading. There are transitional plans for the immediate future, and the design for the Library to be built on the site recently acquired by the University is to include a separate room for the Association's collection, open to members of the Association and adjacent to the section occupied by the mathematical books belonging to the University. The Library of the Association will remain the property of the Association and retain its identity unimpaired ; whether it will develop in combination with the allied section of the University Library into one of the important mathematical libraries of the country is for members to determine, by the use made of it and the gifts made to it : the opportunity has been created.

The Teaching Committee.

The full Committee met in March, and received reports on the work of its seven Sub-Committees.

The Trigonometry Sub-Committee presented the third Draft of Part 1 of the Report on the Teaching of Trigonometry. This was passed for publication, and is now being printed. The Preparatory Schools Sub-Committee presented the Revised Syllabus and Preamble on Mathematics in the Public Schools Common Entrance Examination, which had been prepared at the request of the Headmasters' Conference. There were also some documents from the Technical Schools Sub-Committee.

During the year all the Committees have functioned with energy and enthusiasm. Part 2 of the Trigonometry Report will be ready for consideration early in 1948; the Modern Schools Sub-Committee will have completed an Interim Report on the work in these Schools, while the other Committees all have much material under active consideration.

The work of the Visual-Aids Sub-Committee in preparing the splendid Exhibition at the Annual General Meeting of 1947 has rightly earned the warm appreciation of the whole Association.

All the Association reports have continued to sell at phenomenal rates during the year; the difficulty has been to keep them in print.

The Problem Bureau.

The Bureau has continued to provide solutions at about the usual rate. This branch of the Association's work is especially valuable to teachers in schools where there is only one specialist mathematician. In view of the shortage of entrants to the ranks of specialist mathematical teachers, it is likely that this part of the work will increase rather than diminish.

The questions are still mostly those of scholarship standard, and during the past year the subjects have been almost equally represented.

Officers and Council.

The Council wishes to record its thanks to Prof. G. B. Jeffery for his services as President. Sir Harold Spencer Jones, the Astronomer-Royal, has accepted the Council's invitation to hold the office of President during 1948.

The Council has accepted with regret the resignation of Mr. K. S. Snell from the office of Honorary Treasurer. Mr. Snell has held this office since the Annual Meeting of January 1936, and his skill and energy have been outstanding. Owing to his prudent arrangements, during the war years, the Association now finds itself with a substantial balance with which to face the cost of reopening the activities which had lapsed during the war. The Council in recording the thanks of the whole Association to Mr. Snell, is happy to think that Mr. Snell's advice and counsel will still be available. Mr. J. B. Morgan, who has assisted Mr. Snell during the last year, has been nominated to fill the vacancy.

Mrs. E. Shuttleworth and Mr. F. J. Swan have retired from the Council in rotation, and the Council thanks them for past services rendered.

The Council wishes to record its gratitude to Prof. Broadbent for ably continuing his distinguished work as Editor of the *Mathematical Gazette*. The Council feels that during this year the services rendered to the Association by Prof. E. H. Neville have been outstanding. It is largely due to his untiring work that the very satisfactory solution of the problem of the future of the Library has been solved in so eminently satisfactory a manner, and the Council feels that the whole Association should take note of the very rea

contribution to the future welfare of the Association that has been made by Prof. Neville. The good work done by Mr. Langford, Mr. Daltry and the various sub-committees of the Teaching Committee also merits the special thanks of the Association.

Mr. A. S. Gosset Tanner has just completed twenty years' service as the member in charge of the Problem Bureau. As noted above, this useful, though unobtrusive, section of the Association's work is of great value to certain members. The Council desires to congratulate Mr. Gosset Tanner on his twenty years' service and to thank him and his band of solvers, to whom Mr. R. V. H. Roseveare has recently been added, for their continued service.

The services of Mr. F. W. Kellaway (in charge of the Programme Committee), Mr. A. J. G. May (Branches Committee) and Mr. M. A. Porter (Membership Secretary) have also been considerable. Mr. Porter's labours in bringing the membership list up to date have been very arduous, and the new list when it appears will be almost entirely his work.

The Council also extends its thanks to the Secretaries and to Mr. R. E. Gundry, who has very successfully contended with the voluminous correspondence addressed to Gordon Square.

EXAMINATIONS IN SECONDARY SCHOOLS.

The following memorandum from the Mathematical Association has been sent to the Minister of Education :

" The Mathematical Association devoted a whole morning session at the Annual Meeting, 1948, to a discussion on the recent report of the Secondary School Examinations Council. Very full consideration was given to the effects which the proposals contained in this report would be likely to have upon the organisation and teaching of Mathematics in Secondary Grammar Schools. Some attention was also given to the problems of Secondary Technical and Secondary Modern Schools.

The following resolutions were passed by a meeting consisting of about 250 teachers of Mathematics in the Universities, the Training Colleges and Secondary Schools of all types for boys and girls.

1. This Association is of the opinion that any control exercised in respect of the age at which candidates will be entered for the examination is contrary to the spirit of paragraph 30, and recommends strongly the abolition of any reference to an age limit. (Passed with 5 dissentients.)
2. This Association deplores the proposal to abolish the subsidiary level of the old Higher Certificate Examination and considers that none of the proposed papers would take its place. (Passed with 1 dissentient.)
3. This Association hopes that the interpretation placed upon paragraph 32 will not preclude the possibility of a pupil taking Mathematics at the Ordinary level and taking the same subject at a higher level in a later year. (Passed *nem. con.*)

The Meeting endorsed by acclamation the opinion that the proposed date of 1950 was at least one year too soon. Discussions on the detail of the recommendations by the subject Associations, and the necessary consultations with the Examining Bodies, as well as with the Universities and Professional Bodies, could not be carried out effectively in the time available.

These conclusions are forwarded in the hope that the Minister will give them earnest consideration when making his decisions on the report of the Secondary School Examinations Council. The Association would welcome an opportunity to amplify these conclusions either in a further memorandum or by means of a deputation."

MATHEMATICS AS AN EDUCATIONAL EXPERIENCE.

By G. B. Jeffery, M.A., D.Sc., F.R.S.

Presidential Address to the Mathematical Association, January 1948.

Our Association is happy in including within its membership two intermingling streams, those whose interest lies in mathematics as a great branch of learning, and those who see in the subject the possibility of a great instrument of education. Whether his interest comes from the one stream or the other, no English mathematician could fail to regard it as a signal honour to be invited to serve the Association as its President.

Among the obligations of the President is to prepare and deliver a Presidential Address. In other places, a presidential address may be a declaration of policy and leadership, given in the first flush of accession to presidential power. The President may be tempted, if he can, to make his address a great display of learning so that his hearers may be convinced of the wisdom of their choice of a President and of the folly of questioning his presidential direction. Your custom has wisely ordained otherwise, and your President may speak only after the steps have already been taken that will ensure that within the hour he will become an ex-President.

That custom has a special advantage for the President, for it gives him a clear indication of what the theme of his discourse should be. From the presidential Pisgah he should declare the Promised Land as he sees it, and as he hopes it will be enjoyed under the leadership of other Presidents. And, as the milk and honey are always relative to the hardships of the wilderness, the President on these occasions may claim some moderate right of reminiscence.

It is now more than half a century since somebody first tried to teach me something of things mathematical, and in the time which has since passed a great change has come over the teaching of mathematics in the schools of this country. Age-old traditions have been brought under assessment and have been radically changed. New life has flowed into the teaching of the subject. Honoured names spring to our minds. In private duty bound, I must remember Thomas Percy Nunn—your former President and my teacher and master. You will bring other names to the list, and happily not a few of them will belong to those still with us as honoured members of our Association. New textbooks have been written, which have left their mark on mathematical teaching. A library of the current school textbooks in mathematics written in this country need fear comparison with such a library from no other country in the world.

All this might have had little effect if it had not been for the devotion of the rank and file of teachers of mathematics. I know of no subject in which the teachers have acted under a stronger sense of professional duty and have devoted more time and energy to the improvement of the teaching of their subject. Our Association, especially through its Teaching Committee, has inspired and sustained this great movement and has provided the chief medium for its corporate expression. The Association has a record of which we may justly be proud.

But pride in the past finds its only justification in duty in the present. My recollection of the discussions at your Council during the past year lends little colour to the view that the work of the Association is abating. Much is under consideration in relation to the teaching of mathematics in grammar schools which has perhaps been our main preoccupation in the past. We have our

contribution to make to the burning problem of the modern school curriculum. The persistent questioning of the university curriculum in all its aspects means work for the Association in that field.

Critical times lie ahead for the position of mathematics in the school curriculum. A clearer realisation of the purposes and possibilities of education and a growing knowledge of the course of the development of the young person have broadened our ideas of the kind of experience that a child should gain in school. There is a wealth of new educational material, and the value of the old is called sternly into question. We shall have to face increasing competition from other subjects for the available time in the curriculum. Mathematics is among the " haves " and must expect to attract the envy of the " have nots ". Possession is no longer nine points of the law.

Moreover, mathematics is peculiarly open to attack by the anti-intellectual forces that are rampant in our national educational thought. Some would have us believe that effort and struggle are bad in themselves or at least only good for the very good. It is sought to escape from the stern disciplines that teach us that good things demand hard work for their achievement, and we are offered the alternatives of soft options and sagging standards.

Our current controversies on examinations and on the reorganisation of secondary education are all infected by this heresy. No one can have a long experience of examinations without an acute awareness of their imperfections and of the injustice that they not infrequently work. Yet it would be idle to deny the stern discipline they effect by discriminating between sound knowledge and verbiage and between conscientious preparation and slacking. We can unite in our support of the plans for secondary education for all, and for each the form of secondary education best suited to his needs, yet we may deplore the efforts to secure an easy parity by limiting the possibilities of the best to the capabilities of the weakest.

Every boy has his potentialities which he will never fully achieve. The gap between the possible and the actual will be determined partly by the opportunities we give him but mainly by his own moral qualities of application, perseverance and self-sacrifice. It should be the prime purpose of education to support the boy in the development of those qualities and that is not done by soft options and sagging standards.

It seems to me that once we allow our educational policy to be determined by this heresy we have set our feet on the path of national decadence. I am gravely disquieted by the contrast in this respect between the trends of our own national policy and that of at least some of our continental neighbours.

It is against this background that I ask you to consider the reasons for the high place that mathematics has hitherto occupied in education at all levels and whether we have good grounds for maintaining as mathematicians that we have an undiminished part to play in the education of the future.

I will spend no time in the elaboration of the arguments touching the usefulness of mathematics at every level of the national life or the necessity of maintaining and developing the great body of mathematical knowledge. There must be a due succession of professional mathematicians and, whatever other duties may rest upon our grammar schools, it is upon them that we must chiefly rely to discover mathematical talent and to give it the opportunity to develop.

I am concerned to discover what part, if any, mathematics can usefully play in the education of ordinary boys and girls who are not destined to become professional mathematicians, who are unlikely to use anything beyond elementary arithmetic in their work and their leisure, and who will probably forget most of their formal mathematical knowledge within a few years of leaving school.

I am disposed to make a bold claim for mathematics. It is a body of knowledge now so vast in its extent that the most learned of us can hope to master only patches of it. But it is more than this. It is a way of thinking so fundamentally a part of our human thought that even those who are most proud of their ignorance of mathematics, when thinking of things far remote from the technicalities of mathematics, are often thinking of those things in a mathematical way. In any piece of constructive and intelligent thought, the mathematician will recognise some of the great concepts of his subject. He will find that the thinker is using these concepts in a mathematical way though without employing the technical language of mathematics. If it is possible to imagine that the thinker adopted rules of thought that rigorously excluded everything mathematical he would find that his intellectual power had gone. Mathematics is not only a branch of technical knowledge, it is one of the modes in which the human mind functions. It is especially characteristic of the more developed forms of thought in which the growing young mind must achieve some measure of proficiency. Elementary mathematics affords great scope for the exercise of these forms of thought in a simple way. That, it seems to me, is the educational value of mathematics.

I am well aware that this claim requires for its proper examination a learning in philosophy and psychology to which I have no title and to which I do not pretend. Any value that there may be in what I have to say will be that of a record of the nature of mathematical thought as it appears to one who has spent much time in mathematical studies, who has often pondered the meaning of mathematics, and who has tried to understand the way in which mathematical understanding grows in his own mind and in the minds of his students. It is therefore liable to all the errors of introspection as well as all the pitfalls that beset the path of the amateur philosopher.

I will begin with our domestic cat who, within his somewhat limited range of concerns, is a highly intelligent representative of his species. I often amuse myself by trying to form a picture of what is happening in his cat's mind. I know that the behaviourists warn me that my picture will not be a true picture. Sometimes I catch the cat looking at me in a way that suggests that he agrees with the behaviourist and is chuckling at the thought of how little I know. For what it is worth this is the picture I see : an activity in constant change, usually at a very leisurely tempo but capable of remarkable acceleration. The pattern of that activity is built up from a relatively small number of ideas. There is the idea of fish, to which belong the taste and smell of fish and the prospect of fierce mastication and comfortable repletion. Another idea belongs to a favourite chair with its memories of warmth and comfort. There is the idea of the ginger cat who lives next door. There is the idea of my wife which is strongly associated with the fish idea. There is an idea of myself with an uncertain and tenuous relation to fish but with a strong negative relation to the chair idea since he and I are strong competitors for its enjoyment. There is an idea of mouse which puzzles me, for I have never been able to decide whether it is singular or plural. I doubt whether that cat has ever faced up to the problem of whether the mouse he chased and missed last night was the same mouse or a different mouse from the one he chased and missed the night before, or whether either of them was necessarily different from the one he chased and ate a week ago. He has a few more ideas. There are not very many of them, and they are all immediately related to his perceptual experience, past or present. The pattern of his mind displays these ideas in changing emphasis and in changing relationship. It is not a contemptible mind, for I observe that by it he directs his conduct in such a way as to get most of what he wants most of the time.

If without disrespect to my cat I may take my picture of what goes on in

his mind as indicative of the nature of thought at its most elementary level, I may ask what happens as thought is developed to higher levels. In the first place the range of perceptual experience is vastly increased and is reinforced by second-hand experience. My idea of the Battle of Hastings has probably little correspondence with anything that was seen or heard by anyone at Senlac in 1066, but that is my error. At this level I am still thinking of the external world in terms of the knowledge of it that would have come to me through my senses had I been present at all places and at all times in its history. So far my thought does not differ essentially from that of the cat.

In contrast to the cat mind, I consider my own mind as I am thinking of, shall we say, the theory of functions of a complex variable. I will not suppose that I am attempting to evaluate a troublesome integral because, by a process of which I shall have something to say later, I would in that event probably have come back to the cat level by creating an artificial perceptual experience for myself by making certain marks on paper and then looking at them. Let us suppose rather that I am contemplating the theory and the way in which the assumptions we make as to the small-scale structure of the function condition its large-scale structure and lead, for example, to the majestic generality of Cauchy's theorem. My mind is moving in a world of ideas which is far removed from perceptual experience, and it moves with freedom and with a measure of mastery.

This contrast exhibits the nature of intellectual development. At the lower level the mind is limited to ideas which are directly related to perception and through perception to objects and relations in the external world. At the higher level the mind is liberated into wider fields in which there are many ideas that are not directly derived from perception and which have no necessary correspondence with anything in the external world. The power that comes from such a liberation is easily understood by a mathematician who knows, for example, how much more perspicuous some plane geometry becomes when it is set in a three-dimensional space.

It is interesting to try to trace some of the processes by which this liberation is achieved. An early but important stage is the recognition of similarities between different things—similarities sufficient to support similar judgments. Two dogs are recognised as different and yet for some purposes the same, and a third is immediately recognised on first acquaintance as a bow-wow. The aggregate of the dogs of the child's acquaintance grows by imperceptible stages until it becomes the variety *canis* or perhaps more—the class of all dogs that are or have been or might be.

We all know that the logical analysis of the mathematical concept of class presents problems of considerable difficulty and depth, yet we see this concept emerging in the mind of the young pupil without crisis and developing easily and naturally. It may never attain the subtlety of Bertrand Russell, but it may nevertheless at any particular stage be held with a degree of precision sufficient to support clear thinking.

It is impossible to exaggerate the importance of this development, for it is the necessary antecedent of any form of logical thought. It is also a fruitful source of the proliferation of ideas. Our first notions of class arise from the classification of ideas directly corresponding to perceived objects on the basis of their observed similarities. This implies the analysis of the ideas in terms of abstract qualities. This in turn opens the way for the definition of classes by intention, and the development of logical thinking both in its deductive and inductive phases.

Notice that in the course of this development we have acquired a great stock of ideas which do not correspond directly to perceived objects—ideas of abstract qualities, classes, and the relations of classes. It is the ability to

think in a world of ideas of this order that distinguishes my mind from that of the cat.

One of the great educational values of mathematics is that it affords opportunities for experience in simple thinking of this sort. Our first lessons in geometry may be very concrete and deal with a few triangles drawn on paper or cut from cardboard. We can proceed by easy stages to consider *the* triangle and argue to the conclusion that the sum of the angles of any triangle is two right-angles. The value of the experience is largely lost if the study never passes beyond the concrete, though for a reason to which I shall refer later one must be careful not to get too far from the concrete too soon.

It appears to me that the development of mathematical thought deserves careful and systematic study for the light that it would throw on the development of thought generally. We need a psychological rather than a logical analysis of mathematics, and that is a task I am not competent to undertake. I can only offer the observation that whenever I consider one of the great principles of mathematical thought I can trace its effects on the structure of our general thinking. For example, an aggregate may have properties apart from those of the elements which compose it. It may in particular have unattained bounds or limits. The aggregate of the ideas of all the dots and lines that I could draw on paper is an aggregate in which each element corresponds to a perceived object. But the limits of appropriate sub-aggregates are Euclidean points and lines which have never been perceived. Here again the logically difficult seems to be the psychologically easy, for the intelligent fifth-form boy finds no great difficulty in conceiving the point that is without dimensions and the line that is infinitely thin. I have sometimes tried my younger friends with the argument that if a line gets thinner and thinner, by the time it becomes a Euclidean line there is nothing there at all. Therefore, Euclid is all about nothing. I find that they instinctively reject my logic, though they have not the technical skill to define its fallacy.

This is no fanciful illustration, for it corresponds to something of the deepest significance in our intellectual life. We take our knowledge of the real world as it comes to us through perception and from it we derive conceptions of things that are approached but never attained in the real world. They are ideals. It is when we turn these ideals back again upon the real world so that we set side by side the picture of the world as it is and a picture of the world as it might be that we have the beginning of progress and endeavour. This is the field of the greatest achievements of the human spirit and some of our earliest opportunities of learning to think in this way come through simple mathematics. If the mind in maturity is going to function in this kind of way, the first experiences of this sort are important.

Perhaps the most obvious contribution of mathematics to our general thought is in the field of number, and its associated fields of quantity, measurement, equality, and inequality. Later this afternoon this meeting is to consider this topic from the point of view of the teacher, under the able guidance of Professor Neville, and I shall say less about it than I otherwise would have done. The extent to which these concepts enter into every department of our thought needs no emphasis. It is hardly less obvious that they are necessary for the refinement of logical thought. The quantities of logic recognise only none, at least one, all but at least one, and all. The proposition "Some men are mathematicians" inevitably leads to the question "How many men are mathematicians?", and the reply which leaves us guessing between one and the total male population of the world is somewhat vague. In view of your later discussion I will make only the passing remark that here again the logically difficult appears to be the psychologically easy. Boys and girls acquire without great difficulty an adequate mastery of these mathe-

matical concepts. It often seems to me that the intelligent schoolboy's conception of the cardinal number five is not very far from that implied in the Frege-Russell definition, though he would not understand the terms of that definition. It is, I believe, a good deal nearer to the mark than, for example, the abstraction of the quality of fiveness from all the groups of five concrete objects that he has known.

I want next to refer to what appears to me to be the invariable characteristic of mathematical thought at all levels. It is concerned not with the nature or origin of our ideas but with the way in which our minds function in relation to ideas. If I tackle a new mathematical problem my mind is filled with a complex jumble of loosely related ideas, and I am bemused and uncomfortable. Eventually I discover that I can sort out these ideas and relate them in a pattern in such a way that the whole becomes cogent and significant, and then the problem is solved. Where does the pattern come from? Is it inherent in the ideas and awaiting discovery, or do I supply it from a stock of patterns with which I am familiar and compress the ideas within it with whatever violence may be necessary? Why do I need a pattern at all? Is it because my mind cannot function in relation to more than a certain limited measure of complexity and that I am accordingly in a perpetual struggle after a simplicity that is within my competence? These are questions that I must leave to you unanswered, but I have no doubt that we are effective in our thinking very largely according to the measure in which we are able to attain this tidiness of mind. We can, of course, have superficially tidy minds just as we can have superficially tidy bedrooms, but these constitute no argument against the value of good order as a practical aid to effective work in any sphere.

A great educational value of mathematics is that it affords ample opportunities in a relatively simple field of ideas for the attainment of this tidiness of mind. The mathematician acquires great skill in discerning order in complexity. He can marshal the steps in an algebraic calculation. The product of two projective pencils will stand out to his eye from a complicated diagram. He can order the sequence of propositions to build up a great mathematical system. The mathematician is liable to the failings of his craft. He may exalt tidiness even above godliness. He may, and often does, come to live in an oversimplified world, but he can think effectively in mathematics, and the vast structure of modern mathematics is the measure of what he can achieve.

But have we been talking about things that are essentially mathematical and have relevance only to algebra and geometry? Or have we been talking of things that have their due relevance to straight thinking to whatever end it is pursued? If so, we have one more ground for holding that mathematics, quite apart from its technical content, is an essential part of the structure of our thought. Mathematics, or some activity that provides equivalent opportunities, is a necessary part of the experience in which the young mind develops.

I am not thinking specially of advanced mathematical studies, for the experience may well begin in the nursery school with the orderly arrangement of material objects and the first steps in number work. Young children find a peculiar satisfaction in order and orderliness, and they generally enjoy mathematics until we force their pace beyond their capabilities.

The value of mathematics is enhanced if the search for order is consciously pursued, and if the generalised tidiness of mathematics is exhibited as including ordinary tidiness as an important and particular case. Careless paper work does not assist the pupil in the discovery of mathematical order or in the solution of mathematical problems. A muddled blackboard is a singularly inappropriate medium for the illustration of mathematical ideas.

I have suggested that the development of thought consists very largely in the acquisition of the power to deal with ideas that are not directly related to perception, and that the more developed our thought becomes the more tenuous the relation between our ideas and anything that can be heard or seen. Mathematical genius might be conceived as an escape of the human spirit from the trammels of the flesh into a rarefied space of its own making. Certainly there are mathematical occasions on which any attempt to link thought with perception leads only to confusion. You will not get very far with the study of hyperspace until you give up trying to see what four lines meeting at a point and each perpendicular to the other three look like. But these are the ways of mathematical genius, and for most of us flights into the rarefied atmosphere of abstract thought must be short and we must frequently return to solid ground.

Continued abstract thought for most of us leads to a certain tension that is not entirely due to unfamiliarity. Abstract ideas are apt to be fleeting and difficult to hold before the mind. We seek to anchor them to the more permanent ideas which are in direct relation to perception. Sometimes it is possible to do this in a simple and direct way as, for example, when I draw a graph of a function or make a model of a tetrahedron. Sometimes my thoughts have developed so far away from the perceptual that these simple expedients are no longer possible. Then I relieve the tension by creating a situation which will give rise to a perception which can serve to support my abstract idea. I invent a symbol for my idea and the idea acquires a pseudo-concreteness in the symbol. I feel that I have re-established my relations with the real world, and I am ready to begin the whole process over again. My new beginning is, however, different from my old because my thought is based on symbols and not on perceived objects, and I may forget that they are symbols.

This perpetual effort to externalise our abstract ideas has important consequences. The musician and the painter do more than explore the variety of sense perception. They are thinkers who evolve highly abstract ideas and who have the skill to create sights and sounds which serve them by giving perceptual support to their ideas. To the creator the painting or the symphony is an apt symbol of his thought. Every time he sees it or hears it afresh it brings back his thought, not in the static manner of mechanical reproduction but in the dynamic way of a living and changing thing. The mystery of art is whether and to what extent the work of art induces in our minds something of the thought that was in the mind of its creator—a thought which by its very nature is incapable of representation in material form.

There have been moments when I have been disposed to claim that among the many facets of our subject there is one which shows mathematics as a great medium of art comparable to music. This is not because of any resemblance of structure between a symphony and a mathematical theory ; so far as I can discover there is none. Behind the sounds of the symphony there is the mind of the composer which has conceived things that transcend all sound and sense. He has heard the music that echoes through the temple of the human spirit, and he has sought to catch some of the tones and to fix them in the notes of his score. As the sounds of his symphony fall upon our ears something stirs within us and we begin to hear faint echoes of that greater music which no orchestra can play.

There is something behind the symbols of a great mathematician, something as distinct from those symbols as great music is distinct from the vibrations of air. The mathematician has been with the musician in the temple and each is striving to express in his own medium what he has found there—striving and imperfectly achieving.

There is another aspect of this process of the externalisation of abstract

ideas which is of special significance to us as teachers of mathematics. It is important that we should learn to think abstractly, but most of us have only a limited capacity for sustained abstract thinking. No doubt we learn by practice and experience, but the pace of our learning cannot be forced. Under any attempt to force the pace the mind protects itself by substituting the symbols for the ideas and proceeding to juggle with the symbols. This is a perpetual danger and is present at all stages. It makes the teaching of mathematics a task of great delicacy. If we make our approach crudely practical we rob our pupils of the opportunity of developing their powers of abstract thinking. If we make our approach too theoretical our pupils take refuge in symbol shoving, and the result is much the same. The broad lines of the technique that should be followed are clear though much care is needed in its detailed application. Mathematical thought, like any other kind of thought, must begin close to perception. Accordingly our first approach should be concrete. We should, however, begin to build up abstract ideas as soon as our pupils are ready for them and proceed at a rate that is within their powers. Abstract ideas should be buttressed by a wealth of concrete illustration and example. Symbols should be recognised as symbols and not allowed to obscure the ideas for which they stand.

Perhaps a word may be said in favour of symbol shoving. It is a peculiarly satisfying human experience to attempt a task of some difficulty and to know beyond a peradventure that one has succeeded. The child has many opportunities of this sort on the physical side. He tries to walk and is proud of his early successes. He attempts to vault a bar in the gymnasium and he lands safely and not too clumsily on the other side. He sets out to climb Snowdon and has the satisfaction of sitting on the cairn. In handicraft he has similar opportunities in manual dexterity. But once he has learnt to read he has relatively few opportunities of this sort on the intellectual side. Whatever task you set him in history, geography or French, he can make some kind of a shot at it, but however well he does there is always room for improvement. Failure is seldom absolute and success is never complete. But set him a sum to work or a mathematical problem to solve. He either discovers how to tackle it or he does not. He either gets the right answer or the wrong. He either succeeds or he fails. At a stage at which interests are not developed to the point at which they can sustain long-continued effort, there is a value in these short-term objectives that can be attained or missed within the hour.

But symbol shoving should not be mistaken for mathematics any more than crossword solving (which is of course an intriguing and delightful form of symbol shoving) should be mistaken for literature. That this mistaken identification has often been made is the misfortune of most of our university teaching of mathematics and of that part of our school mathematics which falls under the shadow of the university scholarship. The emphasis is placed almost entirely upon problem solving and there is little attempt to discuss the philosophic nature of the elements with which mathematics deals or to exhibit mathematics as the achievement of great minds. One technique follows upon another and the student frequently gets no more from his study than an amazing facility in the manipulation of symbols.

The time has come when this rambling discourse should draw towards its close. I have been concerned to convince you that as teachers of mathematics you are teachers of a great subject ; that you stand in a long and honourable succession ; and that it is within your power to give something to your pupils that they can receive from no other source. I have striven to find an expression for my own idea of mathematics that will call an answering response from you. The knowledge and the skills that the study of mathematics imparts find many and varied applications and that is sufficient to ensure that

mathematics will continue to hold an important place in the curricula of our schools and colleges. But our ultimate concern is not for the teaching of mathematics but for the training of young minds and the education of young people. We wish for them that they may have full and useful lives and that they may be well-equipped to face all the problems that life may bring them. From whatever point of view we examine the intellectual power that makes man great and capable of great things we find elements within it which are essentially mathematical so that we are impelled to the conclusion that mathematics is part of the very substance and structure of human thought. You will not suppose that I have suggested that mathematics is the whole of that substance and structure and that there are not other elements equally essential and of equal value. But mathematics has the peculiar value that, rightly conceived and rightly taught, it affords the opportunity at its earliest and most elementary stages for the child to think in the ways in which a man must be able to think.

And so, as your President, I would encourage you all in your work in our schools and colleges; and for our Association I would wish that it may long continue to be a power for good in education. G. B. J.

ANNUAL GENERAL MEETING, BIRMINGHAM, 1949.

Arrangements have been made for the next Annual General Meeting of the Association to be held in Birmingham, from 20th to 23rd April, 1949. Through the courtesy of the University of Birmingham, accommodation and meals will be available for members; a notice concerning booking will appear in a future issue of the *Gazette*.

The Programme Committee will be glad to receive, at once, suggestions of topics for discussion, subjects for papers, and possible speakers. These should be sent to F. W. Kellaway, Apsley Day Continuation School, Hemel Hempstead, Herts.

GLEANINGS FAR AND NEAR.

1553. "*I have a passion for solving mathematical problems, and when I have a good problem to work on, and ideas for solving it come into my head, I cannot rest or do anything else until I have tried my ideas out with pencil and paper.*

At the same time, when I am working at a problem I distinctly have the feeling that I am indulging in a slightly forbidden activity.

In Freudian terms, I am sure that my mathematics are a sublimation of some guilty activity, and the sublimation is not quite perfect, so that the guilt seeps through. That this idea is not so strange as it sounds is borne out by the reactions of many people to mathematics."

I have noticed that there are a great many people to whom mathematics is a forbidden subject. They cannot read a scientific book if it contains the simplest mathematical formula, and if you try and explain any mathematics to them they react quite strongly against it.

The history of mathematics bears this out. You have only to study the violent feelings which were evoked about the use of imaginary quantities in algebra, and at an earlier date, at the use of negative quantities.

Your statement that many intelligent people can acquire no interest in mathematics is too mild: you should have said that many intelligent people have an intense fear of, or antipathy to, mathematics.—*News Chronicle*, Advice Bureau. [Per Dr. D. Pedoe.]

A COMEDY OF ERRORS.

By G. N. Watson.

The following question came near to being set in a recent examination :

The area of a triangle is calculated from the formula

$$\triangle^2 = s(s-a)(s-b)(s-c),$$

where $2s = a+b+c$. *It is known that the measurements of the sides are correct to within* 1 *per cent. Find the greatest possible percentage error in the area.*

At the first blush it is tempting to assert that, when the sides are liable to errors not exceeding $a\epsilon$, $b\epsilon$, $c\epsilon$ in absolute value (where ϵ is positive and ϵ^2 is small enough to be neglected), one obviously obtains the largest and smallest possible triangles by altering the linear dimensions of the correct triangle in the ratios $1+\epsilon : 1$ and $1-\epsilon : 1$ respectively, so that the greatest possible error in the area is $2\epsilon\triangle$, either in excess or in defect. These assertions, unlike so many assertions of what is " obvious ", are not completely false ; what is really interesting about them is that, while they are not completely true, they are certainly partially true. Moreover, the formula for the greatest possible error in the area of a triangle when bounds are set on the absolute values of the errors in the lengths of the sides exhibits discontinuities ; one could hardly hope to have a less artificial example of discontinuity in an elementary problem than the example provided by this formula.

The cumulative effect of these statements made it clear that the question was unsuitable for the purposes of an examination, but this very defect made it equally clear that it would be unfortunate if the entertainment to be got out of it were confined to the persons responsible for the composition of the examination paper in which the question was not set.

It is convenient to rewrite the formula quoted above explicitly in terms of the sides, thus

$$16\triangle^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4,$$

and then, with any small errors δa, δb, δc in the lengths of the sides, we see by differentiation that

$$32\triangle \,.\, \delta\triangle = 4(b^2+c^2-a^2)a \,.\, \delta a + 4(c^2+a^2-b^2)b \,.\, \delta b + 4(a^2+b^2-c^2)c \,.\, \delta c.$$

Hence we have immediately

$$4\triangle \,.\, \delta\triangle = abc(\cos A \,.\, \delta a + \cos B \,.\, \delta b + \cos C \,.\, \delta c).$$

Now, when δa, δb, δc do not exceed $a\epsilon$, $b\epsilon$, $c\epsilon$ respectively in absolute value, the greatest possible value of the expression on the right is

$$abc(|\cos A| \,.\, a\epsilon + |\cos B| \,.\, b\epsilon + |\cos C| \,.\, c\epsilon),$$

and the (algebraically) least possible value is

$$-abc(|\cos A| \,.\, a\epsilon + |\cos B| \,.\, b\epsilon + |\cos C| \,.\, c\epsilon).$$

At this point we have to discriminate between acute-angled and obtuse-angled triangles.

When the triangle is *acute-angled*, so that $\cos A$, $\cos B$, $\cos C$ are all positive, the greatest possible value of $4\triangle \,.\, \delta\triangle$ can be written in the form

$$\begin{aligned} &abc\epsilon(a\cos A + b\cos B + c\cos C) \\ &= Rabc\epsilon(\sin 2A + \sin 2B + \sin 2C) \\ &= 4Rabc\epsilon \sin A \sin B \sin C \\ &= \frac{32R\epsilon\triangle^3}{abc} = 8\epsilon\triangle^2. \end{aligned}$$

From this result it is evident that the greatest possible value of $\delta\Delta$ is $2\epsilon\Delta$, and so the (algebraically) least possible value is $-2\epsilon\Delta$. The assertions made above, which were described as "obvious", are consequently true so far as acute-angled triangles are concerned.

Next consider *obtuse-angled* triangles; and, for the sake of definiteness, take the obtuse angle to be A. Since $\cos A$ is now negative, the correct expression to replace

$$abc(|\cos A|\,.\,a\epsilon + |\cos B|\,.\,b\epsilon + |\cos C|\,.\,c\epsilon)$$

is no longer

$$abc\epsilon(a\cos A + b\cos B + c\cos C),$$

but

$$\begin{aligned} abc\epsilon(-a\cos A + b\cos B + c\cos C) &= Rabc\epsilon(-\sin 2A + \sin 2B + \sin 2C) \\ &= 4Rabc\epsilon \sin A \cos B \cos C \\ &= 8\epsilon\Delta^2 \cot B \cot C. \end{aligned}$$

Hence the greatest possible value of $\delta\Delta$ is now $2\epsilon\Delta \cot B \cot C$, and it is easy to see that the (algebraically) least possible value is $-2\epsilon\Delta \cot B \cot C$. It is to be remarked that $\cot B \cot C$ is equal to 1 when A is a right angle, so that the discontinuity in the greatest possible value of $\delta\Delta$ as A passes through the critical value is a discontinuity in form without any actual saltus; also, when A is an obtuse angle (so that $B + C$ is an acute angle), the value of $\cot B \cot C$ is greater than 1.

It is now clear that, for obtuse-angled triangles, what appeared to be "obvious" is false. The answer to the potential examination question is 2 per cent. for acute-angled triangles, but $2 \cot B \cot C$ per cent. for triangles obtuse-angled at A.

A simple graphical representation of the discontinuity is obtainable by taking a horizontal equilateral triangle as triangle of reference and representing on it the triangle with angles A, B, C by the point whose areal coordinates are proportional to A, B, C. At points representing acute-angled triangles set up vertical ordinates of height 2, but at points representing triangles obtuse-angled at A set up vertical ordinates of height $2 \cot B \cot C$; similarly at points representing triangles obtuse-angled at B or C set up ordinates of height $2 \cot C \cot A$ or $2 \cot A \cot B$ respectively. The lengths of the ordinates are consequently proportional to the greatest possible percentage error in the areas of the triangles represented by the feet of the ordinates when the percentage errors in the lengths of the sides of the triangles do not exceed an assigned small number. The locus formed by the upper ends of the ordinates is a surface in the form of a valley with a flat triangular bottom surrounded by three slopes of unlimited height. G. N. W.

1554. The husband who abused his supremacy forfeited half his goods to the wife, and half to the goddess Ceres, and offered a sacrifice (with the remainder ?) to the terrestrial deities. This strange law was either imaginary or transient.—E. Gibbon, *Decline and Fall of the Roman Empire*, Ch. XLIV, f.n. 123. The brackets and the query are Gibbon's. [Per Prof. E. H. Neville.]

1555. In nearly every instance, mathematical methods which are now commonly employed in electrical engineering analyses, originated as academic researches in pure mathematics, and the debt that the engineer owes to the professional mathematician is considerable.—S. Austen Stigant, *Modern electrical engineering mathematics* (Hutchinson, 1947), p. 9.

A UNIFIED COURSE IN MATHEMATICS IN SECONDARY SCHOOLS.*

Mr. A. W. Riley (Wolverhampton): When I commenced to compile the material for what I had to say to-day I found myself in the same difficulty as that to which our President referred in his Address, namely, that compression seemed almost impossible. However, I have an advantage in opening a discussion towards the end of our meeting, in that a good deal of what I intended to say has been said during the past two days, so that had our meeting lasted two or three days more we might possibly have dispensed with this item altogether! Nevertheless, it is pleasing to realise that the subject of our discussion this afternoon is forming a kind of undercurrent to so much of the thought among mathematical teachers at the present time.

Mathematics in schools, as most of us know it, has been planned as mathematics for mathematicians: the outlook has been primarily that of the University. At the University students follow the different subjects in mathematics; at the same time they are capable of synthesising the separate branches into the subject of mathematics. The University outlook has permeated to the schools for two reasons, of which the first seems most important in connection with "secondary education for all". I have in mind what might be called the argument of the "field-marshal's baton". One of the most dangerous sayings that has ever been handed down to following generations—due, I believe, to Napoleon—was that every private had in his knapsack a field-marshal's baton; one of the difficulties of our present educational system is that we have planned education for the future field-marshals instead of the privates—that applies to mathematics no less than to other subjects. This particular doctrine is so firmly held in this democracy of ours that every parent feels that his child is entitled to a field-marshal's education: hence the aim that at all costs the child must have a grammar school education.

The second reason why the University outlook has permeated down through the grammar schools into the secondary modern schools (and it is the secondary modern schools with which I am particularly dealing this afternoon) is connected with the supply of teachers. The trained graduate mathematician who has not obtained a first class degree has perhaps failed to secure appointment in a secondary grammar school; as a second best he has accepted service in the lower status of the senior (now modern) school. In this sphere his professional aspirations have been directed towards promotion to a grammar school; his work has usually followed grammar school lines because he has lacked either the ability or the incentive to tackle the very different problems ready to his hand. In twenty-five years' teaching experience I have found only rare exceptions to this general rule; in spite of the fact that one of the main reasons has been removed by the unification of the Burnham scales, I believe that this statement is still true. Thus, in the schools in which the problem is probably the most difficult, we have less than the best teachers, who often are not in complete sympathy with the outlook of their schools. That problem has to be faced by all administrators and teachers, and also by this Association.

This University outlook, which has divided mathematics into separate subjects, is of quite hoary antiquity, but it also comes down to quite modern times. I suppose, leaving out the war years, we can regard 1935 as fairly modern, and in that year the Board of Education published the pamphlet

* A discussion at the Annual Meeting of the Mathematical Association, on Thursday, 10th April, 1947, the President, Mr. W. F. Bushell, in the chair.

Senior School Mathematics, in which the subject appears still divided into separate branches. I am sure that the Ministry of Education would not take that view to-day, although the Ministry's educational pamphlets are very guarded in their suggestions.

This present discussion arises, in a certain sense, from the discussion we had at the General Meeting of the Association in 1946, when Mr. Swan opened the question of secondary education in the modern school. He took as his text " The child in our midst ", and he quoted with considerable effect—you will have read the record in the *Gazette* for December 1946—that the first need for the child under the new Education Act was a full and satisfying life in the present, a point to which we have not in the past given all the attention we might have done. We have looked rather to a full and satisfying life at some unspecified future date rather than in the present. The establishment of educational principles in relation to the age, ability and aptitude of the child is the new slogan for education at all ages, so that, in whatever type of school the child may be, the unifying principle in mathematics, as in other subjects, must derive from the consideration of the child himself.

I was interested this morning in one of the points made by Mr. Brookes in his talk on statistics—that one of the best features about the teaching of statistics was that it gave guidance on certain matters in outside life. That is rather upside-down. No subject is of any use for teaching the ordinary child unless it has some root in outside life or the day-to-day experience of the child himself. The aims of education and of mathematics I take to be two-fold. The first, utilitarian, aim is the acquisition of that amount of knowledge of arithmetic and geometry which we must have if we are to live full lives as citizens, and about that particular aim I will say practically nothing, as this audience is already well informed on that aspect.

The second aim, to me the more important, I call the ideal aim ; it is not merely the ability to think mathematically, though that may be held by some to be synonymous. I would suggest that the ideal aim is that through mathematics we may achieve an appreciation of the underlying pattern of the universe. I take it that all of us here do get emotional satisfaction from the mathematical pattern of things ; that mathematics is more than just a logical method—that it gives us something of beauty. I feel that is a rather wider aim than was put forward this morning. Therefore, I was horrified to hear it suggested that, of course, the basic processes would be dealt with in the modern schools—*which would get no further* ; the rest was for the exalted people of higher intellectual powers who would have grammar school and, possibly, University education. We shall defeat the objects of the new Education Act if we act upon that view. I am satisfied from my experience with the ordinary child in the modern school that there are many children who, although they will never be able to follow the logical course which is proper for at least the higher parts of the grammar school education, can still appreciate something of the beauty of mathematics. You may think that is impossible : I can only assure you that I have seen it. I am not claiming that such appreciation on the part of children in modern schools is general, but there are children here and there who do have an emotional response to the beauties of mathematics in the same way that some, but not all, derive satisfaction and a vision of the pattern of things from a piece of music or poetry.

I am strongly of the opinion that we must have two aims, of which the second, the ideal aim, ought to be in the forefront of our thought. Perhaps the common metaphor of streams in a school may be rather appropriate here, because the type of course we so often get in the grammar schools up to school certificate, and still more in modern schools, is very like a number of

streams which unfortunately, instead of coalescing into a broad river, peter out into a desert: there is no synthesis into a single subject that can be called mathematics.

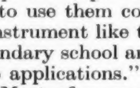
I have said that in pursuit of these aims the unifying principle must be the child himself. It seems to me that we have to consider here three practical aspects which have to be reconciled in working towards the ultimate aims. First, we must consider the basic mathematical ideas: above everything else, I put the idea of functionality. One might say, from the ideal point of view, that for children from 11 to 14 or 15 years of age, the only mathematics worth learning is that which leads on to the calculus, one of the great mathematical ideas which I contend can mean something to many children. Second, in pursuing our course we must make use of the child's interests and experience. I mean here that we must come down to the most mundane topics which are of day-to-day interest to the boy or girl. (If I do not speak specifically about girls it is because my own teaching experience has been mainly with boys.) I am thinking of such things as the football league tables, speed records, or collecting stamps or railway locomotive numbers: we have to use these because they are the things in which the boy is really interested. Then we have to bring in, somehow, the third aspect: the ordinary teaching topics with which we are all familiar: our lesson subjects—percentages, equations, scale drawing, and so forth. We have to get these three aspects unified in one course in *mathematics*. One danger in working purely from the child's interests and experience is that the course may degenerate into a series of snippets—the *Picture Post* or *Weekly Illustrated* type of lesson; those interesting little snippets go in at one ear and out at the other. So we have to try to get a coordinating thread running through our work. I mentioned a moment ago the metaphor of streams: the subjects coalescing at, perhaps, VIth form standard or later. Probably a more satisfying metaphor would be that of a tree, starting with a main stem and branching out at will; in branching out one does not automatically or necessarily cut off the branch from the main stem. We must therefore try to build up a main stem or trunk of mathematics.

As far as the great mass of children in the modern schools are concerned, very little research has been done on mathematics teaching for the reasons I have indicated, in particular the shortage of the right type of teacher. So I put forward what I am going to say about a suggested course, with, I hope, due diffidence. Most of you, I know, are working in secondary grammar schools and what I am saying may appear to have little direct application to your work, but I think it cannot be too strongly reiterated that 70 per cent. of the children in this country will be educated in secondary modern schools and that what happens to them is a matter of general concern. At the two ends of the intelligence distribution curve are two special sets of pupils: those in the grammar school who have had the careful treatment befitting those selected children who will provide the leaders of the community, and at the other end those we call the mentally handicapped who have had special attention because of their handicaps which militate against their achieving ordinary membership of the community. The two ends of the distribution curve have received great attention, but the general average in the middle, the 70 per cent., has not. It seems logical that the general average ought to provide some general rule of treatment and that the other people should be the exceptions; in educational practice we have dealt with the exceptions first, we should now try to work out appropriate treatment for the ordinary child; it is reasonable, therefore, to occupy the time of a gathering such as this with a discussion of this general field in the secondary modern school.

I have had the opportunity of doing a little work in this field, but the scheme I am going to put before you will probably be provocative. I have taught each part of it at one time and another to different groups, but not the whole scheme continuously. You will question the way in which it is linked together, the order of some of the topics ; you will question some of the things I include and some of those I leave out, and no doubt you will often be right. I am going to set before you a rapid summary of a suggested course for the ordinary child in the secondary modern school from the age of 11 to about 13 or 14. I am leaving out of consideration the question whether this course is in any way appropriate to the grammar school child.

I have derived considerable help from Mr. Meredith's paper yesterday afternoon on "Visual Aids". I think we ought to drop the word "aids"—we seemed to be on the point of dropping it yesterday. Mr. Meredith put forward the idea that underlying most of our work there should be what he termed "graphics", and that is the underlying idea of my suggested course. The main interests and experiences of the child are visual and concrete, and the main trunk of our course must also be based upon the visual and concrete ; that means graphs and geometry from the beginning—plenty of drawing and plenty of making things. It means also very little algebra as we ordinarily understand school algebra. Mr. Meredith used the phrase "cartesian revolution", and expressed the opinion that we were perhaps ready for a further revolution now. I would suggest that we had better have the cartesian revolution before we think about the next revolution—I should be glad to think we had really got this cartesian revolution in our teaching of school mathematics. Instead of graphs running through the whole scheme they are often taught as a separate topic belonging to a textbook chapter labelled "Graphs". I want to use them continuously from the beginning, regarding them as a natural instrument like the basic arithmetical processes.

"The pupils entering a secondary school are expected to know their tables and the four rules and simple applications." That is a quotation from the Yorkshire Branch Syllabus. Most of us would, I think, accept it without difficulty. Knowledge of tables should be properly based on conviction derived from personal experience, and applications should have been in fields which have reality for the child—length, weight, capacity, time and money, and possibly area, but not volume. The first need is to ensure that the child has a real grasp of the meaning of number. Yesterday it was mentioned that we must not do anything to interfere with a child's concept of number. The practical difficulty is that we teach children in the primary schools in classes of forty and fifty, and you may have four or five different concepts of number in a class ; often the best we can do is to try to find a common concept which will meet the needs of as many children as possible. This concept should be graphical and should, I think, consist of a length on a scale. I think we tend to be off the track when we show a child 1000 as a cube. It is good to realise how big a thousand or a million is, but the child will not ordinarily be *using* 1000 as a three-dimensional number or 100 as a two-dimensional number. We should reduce each to a length along a scale—the beginning of the graphic idea. I recently asked a teacher : "What do you see in your mind if I give you a number, say 529?" ; he gave me this,



which seems to me rather in the region of the abacus, and quite inadequate

as a symbol for *use*. In working with numbers we must ensure that the child understands the meaning of each separate symbol in the written name of the number and also understands the number as a whole. It is possible for children to do sums and get the right answers without their having the faintest idea what they are about ; this is a process analogous to what our friends the English experts call " barking at print ", in which you see the word and make the right noise without knowing what the word means.

Additional interest can be derived from the history of arithmetic. Thereby the child is led to see that our units, the foot, ounce, acre, and so on, are their particular sizes because each met a specific social need at the time it was invented—in the same way as to-day we talk about light-years and micro-inches. Another very interesting topic will be dealt with later by Mr. Burns, in " The teaching of astronomy in schools ". We may note that a unit with a most respectable antiquity is the degree, as a measure of *time*.

I suppose that in the vocabulary of mathematics one of the words most misused is " average ". I do not know that we commonly deal with averages much in a visual way ; in a grammar school it is probably not necessary to do so ; but I find it helpful in talking about an average to set down scale lines for each item, from which one can see better than from written figures whether the average is likely to have a real significance. You may feel that here we are getting on to the borders of statistics—I hope we are, because I agree most strongly with Mr. Lockwood and Mr. Meredith that statistics are appropriate to school education in mathematics, and that at a very early stage. I have got a histogram from children of 8—using the heights of the members of a class. With care in the selection of the group beforehand a reasonable frequency-distribution curve may be obtained. (I think it was Mr. Parsons who suggested that if you take a number of classes together and get the whole age group of 90 or 100 children you will get a good frequency-distribution curve.) It will not be possible to make much use of the curve at this stage, but we are developing a useful technique, and may see in the growing curve some embryonic idea of functionality. I am not suggesting that one should do this with children of 8, but I am sure that one can usefully do it earlier than the VIth form.

So far I have been talking of numbers, and their relationships will have been brought in when we compare numbers on a scale. One of the ideas we must aim at is that of ratio, which may arise most conveniently from geometrical considerations. There are two ways in which we can introduce geometry. One which many of us have used successfully is " Boy Scout geometry ", based on triangulation, map-making, and so forth—effective because, in these days of hiking and camping, map-reading and map-making are of almost universal interest. But most school geometry is two-dimensional, and the world is three-dimensional. I have derived much enjoyment—and, what is more important, so also have some of my boys—from a three-dimensional approach which has fitted in with my general principle that we must follow a graphical and visual approach. The method is simply a study of the development of the conventional drawing sometimes called " oblique projection ", starting from the drawing of a rectangular block (Fig. 1) in which the oblique lines are at 45° to the horizontal—the method obviously calls for drawing-board, T-square and set-square. From this figure it is possible to learn quite a lot of geometry. If we add a " span roof " to the cuboid we at once come up against the triangle (Fig. 2)—though it must be admitted that the triangle arises less naturally than in the orthodox map-work already referred to. For demonstration purposes I have found figures built up from Meccano strips more effective than blackboard diagrams. For convenience in describing orally the position of points represented in figures in oblique

projection, the insertion of axes and the use of three-dimensional coordinates arise naturally to meet an actual need.

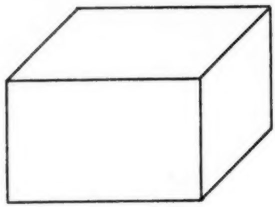


FIG. 1.

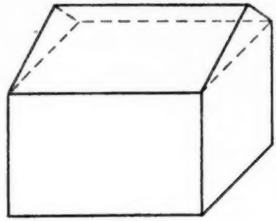


FIG. 2.

I did not know, until the President told us, that Cayley advocated that we should start with geometry in n dimensions and come down later to three and two. I am not sure whether my suggestions are in line with Cayley—one must remember that Euclid, after all, worked from two dimensions upwards!

My boys have had an immense amount of fun out of this geometry, and most of it has been due to suggestions by that great teacher, and former President of this Association, Mr. W. C. Fletcher. As far as I know, there is nothing in print about it.

When the class has gained confidence in using symbols for lengths we can embark on a thoroughgoing course of graphs on fairly orthodox lines, and then turn to map-reading—if not done previously—and we then encounter the idea of ratio, one of the basic ideas I am trying to follow through. With boys of 11 and 12 the graph can be developed up to $y=kx$ or $y=mx+c$. From that I take what has always seemed to me an obvious step: by drawing $y=tx$ for several values of t we build a table of tangents and then do a good deal of work with the tangent without bothering to say we are beginning trigonometry. It always seems to me that the logical time at which to introduce trigonometry is when a boy asks why the angles of a triangle are not proportional to the sides: he is sure to think that they *ought* to be. Then we begin trigonometry and get on to ideas of gradient; in contour work, sections and maps, engineers' gradients are encountered, so introducing the sine. By direct projection the two curves, the tangent and the sine, are drawn. I know the tangent causes some difficulty in the values towards 90°, but for practical purposes you do not deal with gradients approaching 90°.

One topic in which boys are always interested is *speed*—boys generally live in a world of speed. It is too difficult mathematically for the ordinary boy to do very early, but at about this stage you can begin drawing graphs of journeys. Here you get to what a most apt quotation in the *Mathematical Gazette* about twelve months ago described as "the foothills of the calculus". Most boys never get beyond the foothills, but if they get to the foothills they may have a glimpse of the great things beyond, and I do feel that they should have this opportunity. About this stage, still in the early part of the course, work on plotting loci is worth while because of its intrinsic interest; boys are always interested to find how the valve of a bicycle wheel really moves through space.

That completes the main trunk of my suggested scheme. There is no topic in it that I should care to drop, but you will have noted very many that I have left out, some deliberately, because I do not think they should be in;

certain others, like area and volume, are picked up as we go along—I have not mentioned them because their development is fairly obvious. Some of the things I cut out completely, because they are not in line with the child's experience, are the method of " practice ", and " profit and loss ". I am still in two minds about the metric system, because it does not seem to arise out of the child's experience. Other things which might well disappear are the calculation of square root, and a whole lot of algebraic work which occupies unremuneratively such an amount of time, particularly factorisation, which for the modern school child leads nowhere. I am not suggesting that these topics are not appropriate to a grammar school.

There is a danger when we try to make a common course for grammar schools and modern schools that we shall achieve secondary education for all by bringing everybody *down* to a uniform level instead of giving full opportunity for each to rise to his own highest level—that is a danger of which we are all much aware, so that perhaps mention of it is sufficient. Nobody wants to reduce the standard of work being done in grammar schools—in a comprehensive secondary school, surely the grammar school boy, the outstanding *A* boy, will be working in a separate set or form just as he would be inside the grammar school.

The omission of any mention of directed number is probably in your minds. Here again my mind is open as to the stage at which it should be introduced —I find boys in modern schools at 12 or 13 make very heavy weather of it.

This course may occupy two years, probably more. Beyond that, for the ordinary boy mathematics will follow its two trends, the utilitarian, in which case a good deal of what he must know to become a citizen will consist of what we call civic arithmetic. Then there is the ideal aim, but that will apply only to a few. During the first two or three years some will have found the subject attractive : they will continue, others will not. I hope those who continue will do so on the lines of a mathematical club or society. There they will pursue the ideas which seem to them and to the teacher the most fruitful, but I think it no more right to impose mathematics upon *everybody* than it is to impose Latin or music upon everybody. We each get our ideal pattern through some particular channel ; a few will get it through mathematics. I have known just one such mathematical club. The difficulty in developing such work is to find the right man for the job.

I have mentioned the difficulty of finding the necessary supply of teachers, and I want, in closing, to suggest that the function of the Mathematical Association should be to establish and maintain contact with those in the secondary modern schools who are interested but are timid and are mathematically not very well-equipped ; there are many such who would welcome guidance and encouragement. It is necessary to have a good mathematical equipment to tackle independent experiment, and here we can only hope that the training colleges and the universities will help us out in due time. I know personally that many of the emergency training colleges are already doing excellent work in this direction. After all, we are at the beginning in this particular sphere, but I am confident that it is the beginning of something that is very much worth while.

Mr. K. S. Snell (Harrow) : Following Mr. Riley, my task is to present a course suitable for secondary grammar schools. A vital point for discussion is whether it is possible to make this course approximately the same for the first two years, ages 11 to 13, as that for the secondary modern schools, thus facilitating change between the two at the age of 13.

Aim. The objective, or bias, in a course for secondary grammar schools should be *intellectual* study. Appeals to practical examples are necessary to rouse interest, and to give further meaning to students, but utility is not the

ultimate aim. Historically the growth of mathematics has been on an intellectual basis, as, for example, Greek geometry, or the growth of calculus, or the theory of numbers. Practical needs have provided a stimulus, as in the invention of logarithms, but the tendency has been first to theoretical development, secondly to practical applications. Let us therefore keep our examples as close to life as possible, but remember that our aim is to impart enthusiasm for mathematics as an intellectual pursuit.

Unified course. In the last fifty years of teaching we have tended to treat our subject rather like a bunch of apples on a tree, and we have concentrated on the individual apples. Arithmetic was the largest in the initial stages, but algebra and geometry grew as two individual apples, catching up arithmetic, and latterly a fourth, trigonometry, has claimed room for development. But now we see that we should have been putting more emphasis on the tree, mathematics, than on the individual fruits. Let us think, then, of the central trunk. Modern opinion is tending to cut off all branches and leave a severe gaunt-looking trunk, like some city trees just after they have been lopped at the beginning of the winter. But remember that the natural growth is for the branches to break out, and it is unwise to prune too heavily. Thus, while it is a very correct tendency to consider mathematics in its elementary stages as one subject, there does come a time when algebra and geometry require a different technique from that of arithmetic. At this time we must look for every unifying idea, but must not overlook or try to hide the differences which exist.

How then can we lay correct emphasis on unity, while allowing the growth of individual sections within the whole? I would say by having, within the teacher's mind, certain underlying central unifying factors, or aims, which serve as objectives in the different stages, and from which spring the different techniques. These central ideas must be put as objectives, or projects, to the pupils, and will at first be utilitarian. The pupil, as well as the teacher, must know the aim.

As a basis of knowledge, gained in the primary schools, I assume only a knowledge of the fundamental processes of arithmetic. From the beginning of secondary school work I suggest the following sequence, as a possibility, but I must emphasise that the order is only one of many possibilities, and every school must be left to develop its syllabus according to its needs and possibilities.

(1) *Clarity of expression—writing of English.* From the " sums " of early days a child has to learn to solve problems and to write them in intelligible form, problems dealing with cost, speed, simple proportion, etc. He has to learn to write out his work clearly and concisely. He can learn that whereas clear English is needed for some expression, graphical methods can sometimes show results more clearly, and interpretation, and some drawing, of graphs is an essential part of this stage. The use of letters to generalise arithmetical processes, leading to use of simple formulae, will broaden a child's powers of expression, so that in this stage he becomes accustomed to algebraic notation. Ideas of space and solid figures, and a first attempt to sketch figures to make models, and to interpret figures have probably been attempted before this, but they certainly form part of the expression work.

(2*a*) *Measurement and scale-drawing.* Decimals and the metric system must clearly be introduced in connection with measurements. The work should be started with large objects, and should include scale-drawing, preferably at first of objects which the child has measured himself. This introduces the use of instruments and the measurement of angles. It leads to elementary facts about lengths, angles, parallel lines, triangles and the ideas of congruence and similarity. Notice that the geometry branch is here pushing out strongly

and must be allowed to develop. The idea of measurements suggests a numerical approach, but the intellectual development demands simple deductions, simple riders. The first introduction of a trigonometrical ratio, with simple tables in part constructed by the child, follows naturally the work on scale-drawing and similarity. Algebraic notation again arises in the elementary geometry of this stage, and the solution of the simplest equations, *e.g.* in the calculation of angles of a triangle. To the pupil the emphasis must all the time be on increasing power to measure and to draw figures, though theoretical digressions can be encouraged.

(2*b*) *Areas and volumes of figures bounded by straight lines and planes.* This is really a subsection of the work on measurements, and while the emphasis is on rectangles and rectangular solids, the work on areas can also include parallelograms and triangles. Elementary surveying, with measurements made out-of-doors, provides a practical approach to such extensions.

(3) *The inverse process, or working backwards.* Mathematics is full of processes which occur in pairs, addition and subtraction, square and square root, logarithm and exponential, etc. This is the place for the algebra branch to grow more vigorously. Children always find the direct process easier than the inverse. Thus the problem of finding the breadth of a rectangle of given length and area can be tackled directly with the use of algebraic notation. Such problems should not be tackled until the child is thoroughly familiar with the direct process. This leads to the solution of equations and to the inverse use of formulae, at first numerically, and later literally. This must also involve some fractions and their manipulation. Factorisation, first to simplify arithmetical calculations, and then as a mathematical process, is only needed for quite simple factors. It is at this stage that algebra has in the past run riot, with a tendency to heavy manipulation in fractions and factors, through neglect of the central idea of the development of mathematics. It is surprising how little this heavier manipulation is needed in further mathematics.

(4) *Rate, ratio and their application to money.* There is a large section here concerned with ratio, percentage and their application to problems most of which are concerned with money. Examples should naturally be taken from everyday affairs, rates, taxes, etc. But let us beware of trying to thrust down the throats of children of 13 ideas which demand a more mature mind to understand them. Examples can be taken from all branches of civic arithmetic, but I do not think that this can be treated as a subject until boys are nearer the age when they will be directly concerned. The inverse process is here needed, and so this involves the use of algebra. In the measurement of rate graphical methods can be freely used, and the gradient of a graph is automatically introduced and interpreted. Boys, certainly, like to extend this to elementary kinematics, to the finding of velocities and accelerations.

(5) *Simplification of working by the use of tables.* This is a large and important section. Just as over three hundred years ago it was found essential to the advance of navigation to find some means of simplifying the heavy calculations involved, so our children, much more easily defeated by the sight of heavy figures, need to learn to use tables or slide rules before they can tackle the heavier calculations that occur in the mensuration of the circle, etc. They have, we hope, had some introduction to their value in their first treatment of trigonometrical ratios. They now need to learn to use the standard four-figure tables. I find that their need, and value, becomes apparent when, in mensuration, they are dealing with Pythagoras' theorem. This turns them first to tables of squares, the easiest to use, and hence the easiest to teach the use of. At the same time they need the inverse process and learn to use square-root tables, harder but very essential. The use of

logarithmic and trigonometrical tables now follows, and enables the pupil to solve problems of mensuration to a degree of accuracy which he could not before attain, including those problems which had previously been solved by scale-drawing.

(6) *Circle and sphere.* The natural introduction to this section is an extension of mensuration to figures not bounded by straight lines. π can be found experimentally, and the area of the circle can then be deduced by subdivision into small sectors. For a sphere a good start is to consider measurements on the surface of the earth, thus introducing latitude and longitude, and calculations of lengths along great or small circles, the latter with the help of trigonometrical ratios. Also, once interest is kindled in the circle, there is a large and interesting section of geometry involving chords, tangents and angles, with calculations, riders and constructions. But it must be realised that while all this comes conveniently under the heading " circle ", it involves work of quite different types in arithmetic and geometry. Further, the geometry involved is a large section which may well be spread over a year. It is in fact at this stage that the different subjects begin to diverge more widely, though coordination is still possible and desirable.

I have put before you a number of headings under which can be begun the course in mathematics. There is a very great divergence in ability, and in speed of working, of our various pupils. Hence any syllabus of mathematics should be prepared to embrace further applications at any stage.

May I, in finishing, stress that to half of our boys, at any rate in secondary grammar schools, there is a real intellectual delight to be cultivated in the pursuit of mathematics, and it is a subject which they can be brought to enjoy for its own sake.

Miss Barnes (King Edward's High School for Girls, Birmingham): When I was asked to speak at this discussion and was diffident about my ability, I was subtly encouraged by a friend who, after hearing the names of the other two speakers, said : " That's all right, they want a quite ordinary teacher as well." So it is as a quite ordinary teacher that I faced the question of why we coordinate at all, and the answer—obviously because we are concerned with mathematics and for the child's greater understanding, not only of the subject-matter, but of the nature of the subject : not primarily mathematics for the mathematician (though our reward is great when we find one), but mathematics as a development of human thought, with something of its history indicated " by the wayside ", and also (this is the real thing) the development of the child's own mathematical power.

It is a first step forward for the child, drawing rectangles on squared paper to answer the question, " Do you need to count the squares? " with a confident " No! "—a much more advanced step for the thirteen-year-old, using logarithms for the multiplication of numbers, who asked : " And then can we use logarithms to multiply logarithms? "

In the first two years a child in a grammar school will have acquired the use of algebraic notation to give a language and a tool for handling problems, elementary ideas of proportion and mensuration ; the difference between similar and congruent figures ; and with the growth of mathematical ideas there should be a fairly equal growth of technique, fostered by lots of easy examples.

Algebraic factorisation is comparable with multiplication tables, neither are very fashionable, and an easy ability to use both is essential for further work. So I include factors of easy quadratic expressions before the work on mensuration leads to Pythagoras' theorem with its application to areas and numbers, with the necessity for finding square roots (and an equal necessity for finding a quick way of getting them, *i.e.* from tables). One could epitomise

much of elementary mathematics with Pythagoras' theorem adequately discussed.

Mr. Riley omitted directed number from his preliminary course: for me the number scale is really the beginning of graphs. The early work on graphs is mainly representational, including graphs of squares, cubes, reciprocals, but not $y=mx$, or $y=mx+c$, and in the third year there are two main pieces of work: the development of graphical work and the beginning of formal deduction with the group of circle theorems.

Much of the work in the later years of the course needs tackling a block at a time, but rather like the *Second Geometry Report* diagram of Stage A, Stage B, Stage C, so that after the bulk of one piece of new work has been tackled the next can be begun and coordination continued. Very short and very easy oral work at the beginning of each lesson will collect up ideas before continuing, and these should always be easy enough to give confidence.

The preliminary graphical work—graphs of squares, cubes, reciprocals, 10^x, leads on the one side to the use of tables (and particularly the use of log. tables) as tools for easier calculation. On the other side, the work continues with graphs of quadratic functions. It is always a delight to find how much a girl will pick out for herself—maximum or minimum values, the shape of the curve, and (in the solution of quadratic equations by graphical methods) the fact about the sum of the roots and the connection between the idea of a tangent as a limit and the special case of the quadratic equation with two equal roots.

I find it worth while here showing the "completing-the-square" method (the quadratic equation with no roots will have been met graphically), and to show where the method breaks down for this type of equation. It depends on the class how much we do here: straight line graphs with $y=x^2$ to give solutions of any quadratic equation, with $y=x^3$ for solution of $x^3+ax+b=0$; solution of simultaneous quadratics.

Towards the end of this work on graphs a block of work on loci revises earlier ideas, connects up the formal geometrical work with the graphical work, so that we can look back to see the way we have come with functional notation and the algebraic expression of locus.

It is fun and always worth doing to include conic sections as loci. After all, the class should know that seventeenth-century calculations were getting too heavy when Napier and Briggs gave us logarithm tables; should certainly have heard of Newton and have some idea of the problems he tackled; should draw, *e.g.* the parabola as a graph, as focus-directrix locus, as an envelope.

I think it important to be able to realise that a fact is true or it is not; to realise that it is quite surprisingly difficult to make an ordinary statement, the converse of which is true, and I think the formal proofs of the converses of the circle theorems are worth discussing as a separate piece of work, with revision of other facts considered in relation to possible converses.

If pure geometry is considered to have any value at all (and I think it has), then the foundations must be properly laid for the next year's work on ratio. The starting-point is the proportion theorem, formally proved by area facts, of the truth of which all should be convinced, and the three main streams from this are: (1) ratio and variation—the difficulty here is often with the language; (2) work on similar triangles to the areas and volumes of similar figures, with correspondence with variation; (3) trigonometrical ratios, which need their own special practice, and the trigonometrical equivalent of similar triangle facts which is the sine formula.

The difficulty in this work is generally not in being unable to see the pattern, but in weak manipulation, which makes a case for earlier practice.

But the pattern is clear—the graphical representation of variation, the idea

of slope or gradient from contours and three-dimensional problems, the use of similar triangles to get the gradient of a chord, to get average speed, and so (in the limit)—and the suggestion always comes from the class—to the gradient of a tangent, the speed at an instant, and from this to rates of change and maxima and minima.

Here I am feeling my own way with a slow division in its fourth year (a good one takes this in its stride) beginning with the idea of gradient, and I am interested and pleased to see how much follows easily for them and gives an extension of the earlier work on quadratic functions.

If this work is tackled at all, then I think that it should be continued to the evaluation of simple areas and simple volumes of revolution. It is particularly satisfying to show that the reverse process to differentiating (whether you use the word integration or not) has a meaning in its own right!

There are some separate items which need time to themselves, the extension of area work with the cosine formula, the numerical solution of triangles, and I find it very worth while to give time to the drawing of trigonometrical graphs, continued for angles greater than 360°. They give a demonstration of simple wave motion, and it is not difficult to plot or sketch two wave motions superimposed.

How much or how little to do can only depend on the capacity and interest of each girl, and the inevitable time lag between the first excitement of a new idea and the ability to use it easily. But whatever the length of course, there should be some time for looking back, for deliberately stressing the unity of ideas, with different techniques.

For mathematics is a way of thinking, and should be so presented that however short her course a child should have not only confidence in her own powers, but some vision of its further scope and possibilities.

The President, in inviting general discussion, hoped that it would achieve the object, namely, that it would elicit the maximum number of opinions from members of the Association.

Mr. C. V. Durell (late Winchester College) said when he had read the title of the subject under discussion he had thought, as had probably many other members, that it might mean that they had to get their pupils not to think any longer of algebra, trigonometry or geometry or calculus as separate subjects, but now from what he had so far heard he did not think that was suggested. It was essential from the point of view of building up a general picture of mathematics to distinguish between the processes of mathematics and its applications. As long as they were employing the processes of mathematics, they were dealing with algebra or with geometry or with trigonometry or with calculus. Processes possessed the characteristics which gave the names to the separate subjects. When the ability to perform a process, or set of processes, had been acquired, it became possible to deal with mathematics as a whole. That was what seemed to the speaker to be important. It was necessary to get into the minds of pupils that they had at their disposal a number of different weapons: the arithmetical weapon, the algebraic weapon, the geometrical weapon, the trigonometrical weapon. Whatever course the pupil followed it should equip him with at least these four weapons and perhaps also give him some idea of the calculus weapon. The pupil ought to have these four weapons in his tool-bag, and like the plumber he should always keep these weapons at hand, because when he went into the classroom or the workshop they were the weapons which he needed for use there. The teacher's job was to give the pupil the power and discrimination to select the proper weapon for the particular task in hand. When tackling an application of mathematics, it is unnecessary for the pupil to deal with it with each one of the weapons in turn; the teacher's aim is to train the pupil to look

into his tool-bag and select the particular weapon—the hammer or chisel, whatever it might be—best suited to the particular application in everyday life. But the pupil cannot acquire that ability to choose his tools, to know which is the best tool, unless he has thoroughly learned how to use his tools. The only way in which that could be done was by intensive practice and, therefore, he had to have his special practice in a particular process; for example, by learning how to use a ruler and a pair of compasses, or by learning how to use the sine or the tangent of an angle, or how to use tables, to mention only a few of the various weapons, all of which the pupil must be able to use with facility. That could only be done by concentration on the particular task in hand; and that was why one had to have some drill. Indeed, it would not be possible to progress unless there was willingness still to have some drill in the mathematical work. When the pupil had learned how to use a group of weapons, the time had come to look at mathematics as a whole in the scheme of life, the way in which to apply it to life in general, and then to train the pupil to choose which weapon he had to use in order to tackle his job. The later stages of geometry offered ideal fields for practice in selection. For example, in the general discussion of Pythagoras' theorem, the ideas should be discussed arithmetically, algebraically and trigonometrically; but in any particular associated problem, the pupil must be able to make his own choice as to the best weapon to employ, although it might then be advantageous to show him how other weapons could be used. In this way, his sense of mathematics is unified in the field covered by his training.

Mr. R. D. L. Moore (Birmingham Emergency Training College) expressed appreciation of Mr. Riley's helpful remarks. Would Mr. Riley consider adding to his list of omissions H.C.F. and L.C.M. and a good deal of work on prime factors; also the postponement of division of fractions? On both of those the speaker had seen a good deal of time spent to no purpose in secondary modern schools. He would be in favour of confronting pupils with a problem of everyday life and then, if a particular process was needed to solve it, that would be the time to bring it in. Why decimalisation if it was not going to be necessary or would not make the problem easier of solution? Division of fractions did not come in very much, except in problems of area, until later stages and might well be postponed until really necessary.

Mr. D. B. Scott (University of Aberdeen) said it had been a revelation to him, and he thought also to some of his academic colleagues, to find how much emphasis was being laid in the schools on the understanding of mathematical principles. Some might have been particularly unfortunate in their pupils; many of them might not have had the benefit of all the aids that had been mentioned during the discussion. Mr. Scott added that he wanted to get down to what might be termed fundamentals; he had no ambitions to go into the foundations of mathematics, but he wished to mention one or two points which had been overlooked by the principal speakers, probably because they thought them so elementary that all present would take the points for granted. It was necessary to bear in mind that in the last resort any education that might be given would be judged not so much by the effect on the children, but more by the adults into which those children grew. Far too much emphasis might be placed on the intellectual understanding of mathematics by the child, and far too little on the necessary equipment of the non-mathematical adult. Perhaps he might, in the absence of his wife—and maybe she would forgive him for so doing—take her as an illustration of the point he had in mind. His wife was a comparatively well-educated person to whom Cambridge University, but for the accident of sex, would have granted a first-class degree. Her mathematical equipment was, on the whole, rather

sketchy. Of one or two subtle matters she knew a certain amount : she was very good at weights and measures (she was a most excellent cook). She knew a certain amount of various forms of geometry ; she could handle her tape measure ; if taken to a roll of material she looked at the width and knew exactly how much she would need to buy in order to make a dress, or whatever it might be. Her geometrical sense was, on the whole, rather well-developed, but her arithmetical grounding was deplorable.

It seemed most essential to realise that before getting into the minds of pupils any ideas of the basic principles of mathematics—and that must be done—they must be able to do mental arithmetic. That could not be over-stressed. It seemed an absolute scandal that every shop assistant had to count the change out into one's hand. At any rate, the speaker's mental arithmetic was good enough for him to be completely put off when that trick was played upon him, though immediately he looked at the money he could see whether or not it was right. Too many adults who were supposed to have a general education and a grasp of quite abstract ideas of mathematics had not the necessary social ability which was involved in counting change.

There was much talk of logarithms, how to use tables of inverses, squares, and so on, but it was even more important that people should learn how to use a ready-reckoner, not necessarily at an early stage in the school course, but perhaps during their last month at school. The ability to use a ready-reckoner was far more important in many respects than ability to understand logarithms. At some stage during the war his wife had been employed in the occupation known as "Cooper's Snooper", or officially, "Regional Interviewer for the War-time Social Survey". She had, on one survey, been sent out to ask people how much coal they burned in a year and how much it cost them. They replied that they bought a bag a week during the winter, and that it cost them so much a hundredweight. And then his wife came home and asked him to work out all she had put down. That came to be rather a bore and so he had bought her a ready-reckoner, but he discovered that that was not such a clever thing to have done after all, because it was much less trouble for him to work the sum out than to show her how to use the ready-reckoner. Therefore it seemed really necessary to maintain some sense of proportion in approaching school mathematics. The understanding of mathematical principles in itself was admirable, but that must not be achieved at the expense of the essentials of living, and he would put in the forefront mental arithmetic and the use of the ready-reckoner, at least as far as the non-mathematician was concerned.

Mr. J. C. Skinner (Bradford Education Office) had been pleased to hear Mr. Riley give so much prominence to the question of the course in the secondary modern schools, because obviously, with the raising of the school-leaving age, that was a matter of primary importance. Mr. Riley had, perhaps, somewhat over-rated the ability that there would be generally in the modern schools ; he had talked, if he would allow it to be said, rather of the A stream than the B's and C's. In what Mr. Durell said as to the necessity for drill he seemed to be thinking particularly of the grammar school, but in the modern school it did seem that it would be necessary to decide what were the particular tools the pupils should have sharpened when they left school ; they should at least have mastered those tools, and they would not be very many, certainly fewer than was generally supposed. There had been a number of researches into the question of the processes used in ordinary life, and the general result had produced a rather surprisingly meagre selection. There would be certain things to be mastered, certain processes that could be done, indeed that a boy must be able to do as a "producer" of mathematics. There would also be a further slightly larger field of which he should have an understanding, at least to some extent : such things as the distinction between

simple interest and compound interest. It would not be possible to make the majority of leavers from a modern school safe and quick in calculating, or in using tables to calculate, the difference between simple and compound interest on an amount of money over a number of years, but they should understand that there was a difference and be able to refer to someone who could calculate its amount, if only as a protection against the blandishments of insurance agents. It was good to hear the emphasis placed by almost all the speakers on the possibility and desirability of aesthetic and historical interest in mathematics. Even the child of moderate intelligence might enjoy mathematics. As fundamentals there should always be in mind (i) the operations that the leaver from the modern school should be able to *do* pretty accurately and readily, (ii) the mathematical terms and explanations that he should be able to *follow*, and there a good deal of reading of graphs and statistics would be required, a greater degree of ability to read graphs and statistics than of ability to make graphs and compile statistics.

Mr. J. W. Ashley Smith (Henry Smith School, Hartlepool) recalled that Mr. Riley had said that one of the objects in the teaching of mathematics was that the pupils would achieve appreciation of the underlying pattern of the Universe. He thought that a pattern was something which involved an idea of subdivision into well-defined units ; that it would be better if the pupil knew, when he was doing something by any process in mathematics, that it was something he would find in an algebra textbook in the second chapter rather than something he would find in the thirteenth chapter of his General Mathematics textbook. It was, of course, right and sensible to use whatever came to hand in whatever period one was teaching, that when trigonometry came naturally into algebra it should be used, and any who taught in a sensible manner did not refuse to use algebraic methods in arithmetic. Why they should be bound to use a book labelled "General Mathematics" and refuse to tell pupils what subject they were doing, the speaker really could not understand.

Mr. Randall (Royal Grammar School, Lancaster) spoke as one who had returned to teaching—and as a result of being away for some years one saw things more clearly. He felt that as a first stage towards any unification, supposing one found it not very practicable to put into operation at once the wide scheme suggested by Mr. Riley, there should be an endeavour to unify algebra and arithmetic for a start. He had personally been presented with a great problem when dealing with algebra, in pointing out to pupils who had done the subject for three or four years that letters did stand for numbers and that algebra was not entirely a separate subject from arithmetic ; that it had its roots in arithmetic and was not in fact a separate system of mathematics. As a first stage in unification the speaker said he would be glad to see a textbook which did similar processes in the two subjects simultaneously. For instance, why do fractions in algebra come a long time after they have been done in arithmetic, so that they are taught as something entirely different? Being taught out of different textbooks and by different methods, the boy of not very high intelligence did not realise that they were absolutely identical. In regard to most of the textbooks he had seen, he could make that complaint. For instance, there was a tendency to keep a chapter on "Formulae" to some considerable distance on in the algebra course, whereas Mr. Randall could not help feeling that its proper place was near the beginning of algebra when one would be doing similar things in arithmetic, so that one could do the two things simultaneously. Admittedly, there were branches in arithmetic and algebra which had not a corresponding branch in the other subject. It was helpful to try to keep arithmetic and algebra—leaving geometry for the moment—to a large extent unified in this way. A

boy should not know whether he was doing algebra or arithmetic ; he would be dealing with numbers, whether they were stated explicitly as numbers, or as letters which after all stood for numbers in all elementary algebra. There should be no distinction between the two. Much of the trouble he was experiencing with the algebra of boys in the school certificate—or subsequent—year was probably due to the fact that it had not been sufficiently pointed out to them early on that the two subjects were the same. At any rate, the speaker hoped to see a textbook which did algebra and arithmetic together; because they were inseparable and not two subjects.

Mr. W. O. Storer (Highgate School) felt he should speak in defence of a distinguished member present who had included in one textbook algebraic and arithmetical processes. The speaker agreed that much in algebra was of the same nature as the processes of arithmetic, and it seemed thoroughly sensible to learn the processes with the more familiar numbers and then later to go on to the generalisation included in literal expressions. In dealing with algebraic processes, the book the speaker had in mind followed his own practice of giving examples with numbers first, then relating these familiar processes to less familiar objects, and so proceeding to generalisation.

Mr. G. L. Parsons (Merchant Taylors' School) thought the question of the unification of the syllabus fell into three quite distinct categories : first, there was the unification of mathematics which might or might not exist in the mind of the teacher. All present hoped that it existed in their minds, but there was plenty of evidence to show that it did not exist in the minds of all teachers of mathematics, so that perhaps the first effort should be to persuade all teachers of mathematics that such a unification ought so to exist. Secondly, there was the question, less pressing perhaps, as to whether that concept of unification of the mathematical syllabus was one which should be passed on to the pupil and, if so, whether it should be passed on immediately or whether it should be developed stage by stage until the pupil found it out, perhaps with a blinding sense of discovery, somewhere towards the end of his mathematical career. Thirdly, there was what the speaker admitted that he personally regarded as the relatively unimportant question as to whether the unification should or should not be expressed in the publication of a textbook. It seemed that if the unification existed in the mind of the teacher there was some hope that it might eventually exist in the mind of the pupil, and that if it existed in both their minds it did not matter whether or not it existed in a textbook.

Miss Randall (Rugby High School for Girls) had been experimenting during the last term with a more unified course in the case of her second year pupils and had come up against considerable difficulty with the algebra which she definitely wanted to do while not wanting to get too much out of line with what was being done in parallel forms. She had had to drag a good deal of it in artificially, but there appeared to be justification because she knew the work was such that the children enjoyed doing it. When they had done the work they knew whether it was right or not and that gave them confidence—an attitude of mind which it was most important to develop. Miss Randall hoped that others would express their views on the question of the unified mathematics syllabus and the bringing in of the algebra, which most speakers had rather relegated as a later development.

The speaker's second point arose out of Mr. Snell's remarks in regard to work on latitude and longitude, which suggested that it came in perhaps at the end of the second year. By that time latitude and longitude would have been dealt with in the geographical department quite often without giving the pupils any very clear ideas as to where those angles really were. It seemed that that was something the mathematician could well do and in a very

early part of the geometrical course, again making unification not only of the mathematical syllabus but of the mathematical work essential in other subjects.

Mr. S. Inman (Isleworth County School) thought it would be possible to go a step further than unification in mathematics by having unification in education as a whole. There were many points of contact between mathematics and many other subjects. For instance, in mathematics and science there were numerous topics which centred round graphs and formulae ; there had been mention of latitude and longitude ; there were also other points of contact in geography. Descriptive geometry had connections with handicraft work, and perhaps in some cases even the art lessons. There were, no doubt, other points of contact with many other subjects. At the same time, members should not think of having one subject, simply education. Although there were all those points of contact there were the very broad streams, and it was much more convenient and best if those broad streams were dealt with separately. In spite of the many points of contact between different branches of mathematics, it seemed, on the whole, best to treat the main branches of mathematics, geometry, algebra and so on, as separate streams, and where appropriate one could deal with any particular topic with the tool, as suggested by Mr. Durell, that seemed most suitable. It was easier, from the point of view of learning and developing, to have the subject taught systematically by means of the main streams.

Mr. H. V. Lowry (Woolwich Polytechnic) thought a large part of the discussion had centred round unification in the grammar school. A rather more important part was that of unification in the secondary modern school. His feeling, as a result of testing candidates for the A.T.C., was that although they had forgotten most of their arithmetic, in fact their entry standard was deplorable, one could in a short time, because of the interest they had in the A.T.C. work, revise their arithmetic and get them to do quite good work with it. When a boy went to a secondary modern school it seemed that the first thing one had to do was to give him applications in civics and other things which would bring out the arithmetic he had done at a primary school. The unification ought to come by bringing in algebra and trigonometry as they were wanted in the applications. Algebra and trigonometry as such ought not to be taught at all at a secondary modern school ; they could come in accidentally as the pupils tackled problems involving arithmetic ; after that, trigonometry and algebra would give short cuts.

Miss M. J. Meetham (Abbotsholme) noted that both of the previous speakers who had mentioned girls had referred to their lack of confidence in mathematics. She wondered if the time had come to cease pretending that girls did need the same mathematics as boys, both in modern schools and in grammar schools? She found girls short in confidence in mathematics because they lacked interest in gadgets and machines such as boys had, in place of which girls had a more natural interest in people. Girls could thoroughly appreciate the aesthetic beauty of mathematics and master its skills, which were the essentials of living as citizens, but their practical applications, the uses to which they put their mathematics in class work, should be entirely different from those of boys.

Mr. T. C. Batten (Sutton County Grammar School) agreed with Mr. Durell that it was essential that children should first learn the principles which underlay the various subjects included in mathematics. Every teacher had to select what processes he was going to teach, and had to decide in his own mind in what order he was going to teach them. So long as there was a natural flow and the teacher could find at any stage the work he was going to do, what that stage was would not matter very much as long as he was

convinced that he was teaching the right things and had them in the right order.

Mr. I. R. Vesselo (Stationers' Co. School, Hornsey) thought the discussion had centred round the two extreme points of the pupil range. On the one hand, there were those who were making a tentative approach to mathematics in the modern school, putting their feet forward gently and treading warily. At the other extreme there were perhaps the bulk of the members present. They were extremist in sympathy, people interested only in the production of mathematicians. Between these two ranges there was a very large and important group of pupils which was being neglected. It consisted of 60 per cent., or perhaps more, of the pupils at the average secondary grammar school who had been forced in by the dictates of social prestige and who found themselves put through a mill in which they were taught, as Mr. Durell had put it, to use a chisel and went on for several years learning how to use a chisel. Then that would be put in the tool box because there would be no need to use it any longer, and the pupils spent the next year learning how to use a hammer, and so on. The result was that by the end of their school career they had learned how to use two or three tools and had never caught the faintest glimpse of the finished model. It was those pupils whose school-life was being flung away by the specialist, who closed his eyes to the fact that only a very small group of his pupils had any hope of proceeding with the work and who could not be bothered about the rest; he was sorry, but he had no time to waste on them.

It was time that the myth of utility was exploded. Mr. Riley had mentioned utility and Mr. Scott had described the sum total of the utilitarian mathematics required by the average citizen—the ability to count change, and the use of a ready-reckoner; to this might be added the ability to distinguish between odd and even numbers on either side of a road. Those who, like Mr. Lowry, had met the type of student who was interested in mathematics from a utilitarian point of view would know what rapid strides such students made. They would agree that, within a year, starting from zero, the average pupil who sought mathematics for utilitarian ends could outstrip the vast majority of the normal secondary grammar school pupils. It was customary, in National Certificate courses, to exempt those who had reached matriculation standard from the first year of the course; but there was always doubt as to whether it was justified. Those apprentices who had no mathematical background, who had simply come to the class with the desire to learn something which would be useful to them, went far ahead of the grammar school boy in a year.

It was clear that the vast majority of the pupils at the average grammar school would not and could not derive any benefit from drill in algebraic manipulation and other academic work which many members seemed to regard as so important, and rightly, to the mathematician. It was time that something was done for the other pupils in the way of providing them with a picture of what it was all about. The unified course was an attempt to solve that problem. Having tried it in a first form at the beginning of the year, he recognised the difficulties, and they probably accounted for the tendency to move away from it. The chief difficulty was that of textbooks. It was no use setting work as mathematics because pupils asked whether it was in the algebra book, the geometry book or the arithmetic book. Those difficulties could be overcome, and should be, if people were not to be condemned to an intense dislike of a subject without having seen anything of it.

Miss D. Lloyd (Lowther College, Abergele) referred to a successful experiment in unification she had made some years ago, during the first two years of a four-year School Certificate course.

Miss E. Barnes, in replying, expressed the view that too much of the discussion had ranged round textbooks. She had never yet heard of a conference of teachers of English discussing whether there ought to be one general English book so that children might not have to read separately. At the beginning she preferred separate textbooks, and later she might prefer a general one.

It did not matter what the course was called—general, or by separate subject names. As Mr. Parsons had said, the unity of mathematics should be in the mind of the teacher and (as she hoped) in the mind of the child : nothing else really mattered.

Mr. K. S. Snell said he had a list of tools made out but he had omitted them as something with which all present would be familiar. If tools were practised in the first instance, then teachers tended to forget the objective they must have in mind. If, on the other hand, teachers emphasised the second point, the objective to the pupil, then, as the tools were required, the pupil would practise them and be drilled in them much more readily. Therefore, Mr. Snell said, emphasise, above all, the objective to the pupil and let the rest follow.

Mr. A. W. Riley feared he had overshot his time to such an extent that he skipped much of what he had hoped would be taken for granted. Mr. Moore had made a point about H.C.F. and L.C.M., but Mr. Riley was not sure that he agreed as to the division of fractions also being omitted. Mr. Skinner's point rather tied up with that made by Mr. Vesselo as to 60 per cent. of the pupils in grammar schools not getting anywhere, let alone getting enjoyment. When speaking as to teaching in the secondary modern school, Mr. Riley said he had in mind the type of child who *ought* to be receiving education in a secondary modern school—he personally felt that quite a lot of that type of child was in grammar schools at the moment. That was an opinion which was probably not shared by many others. In reply to Mr. Skinner, Mr. Riley expressed the opinion that the ordinary child of the secondary modern school was capable of a good deal more than teachers were apt to give him credit for, when the matter was presented in a guise which the pupil could understand and enjoy. Such a pupil would probably not understand much about the difference between simple and compound interest, but something about them would occur in the civic arithmetic later on.

There were two speakers who apparently disagreed completely with the unified course, and with them most of the other speakers disagreed. Mr. Snell had already made the most important point which arose out of the remarks made by Mr. Vesselo, and particularly out of Mr. Durell's remarks at the beginning, which seemed to be the correct summary of his (Mr. Riley's) own rather rambling remarks. Before one started using a tool, it was necessary to have something for it to do : the tool arose after the need for it was felt. The point he had tried to make was that for all pupils the start should be from the child's own experience. A boy would never be happy if he was being taught to shape a tool when he had no idea as to what the tool could be used for. Mr. Riley hoped Mr. Snell got that point across ; at any rate, it was the point he wished to leave with the meeting.

The President felt all would wish to thank the three speakers who had taken such trouble to get together and present the facts of the case ; they had given the profundity of their wisdom in regard to the unified course and there was an indication that a great deal more would be done about it in the future. That was of the greatest importance.

THE ROYAL SOCIETY.

Scientific Information Conference, 21 June–2 July, 1948.

Arising out of the Royal Society Empire Scientific Conference and the British Commonwealth Scientific Official Conference the Council of the Royal Society has decided to arrange a Conference on Scientific Information Services, to be held in London from 21st June to 2nd July, 1948. Admission to the Conference will be by ticket only, but organisations and individuals interested can make application in writing for tickets to the Assistant Secretary, The Royal Society, Burlington House, London, W. 1., before 1st June, 1948.

CORRIGENDA.

No. 296 (October, 1947). Note 2001, p. 254, l. 6 from end of Note.

For OL $= r$ *read* OL $= 2r$.

No. 297 (December, 1947). P. 257. The obituary notices of Rawdon Levett in Vol. XI were by C. H. P. Mayo and C. Godfrey.

No. 297 (December, 1947). "Determinants and Permutations", p. 284. The authors wish to thank Mrs. Margaret J. Moore for pointing out an error. The statements that if the "quotient" of two permutations contains no chain of more than two links, either both permutations are self-conjugate or both are non-self-conjugate, and that if the "quotient" contains a chain of more than two links the two permutations differ in their relation to their conjugates, are incorrect and should be deleted.

1556. From a script :

Data $\qquad \dfrac{a+b}{b+c} = \dfrac{c+d}{d+a}$

To prove that $\qquad a = c \text{ or } a+b+c+d = 0.$

Proof

$$\frac{a+b}{b+c} = \frac{c+d}{d+a} = \frac{a+b+c+d}{a+b+c+d} = 1$$

$$\therefore a = c.$$

If $a \neq c$, then $a+b \neq b+c$ and $c+d \neq d+a$

$$\therefore a+b+c+d \neq a+b+c+d$$

which is not true unless

$$a+b+c+d=0.$$

[Per Rev. S. H. Clarke.]

1557. Societies, in order to live, had to evolve some centripetal force to hold in check the centrifugal tendencies of their too, too spirited individuals. —Salvador de Madariaga, B.B.C. Third Programme ; *The Listener*, October 10, 1946. [Per Mr. A. J. G. May.]

1558. There were three separate aspects of these recent happenings he had to reconcile, three angles to be resolved before this triangle of a problem could be set up four square and firm for a reasonable solution.—E. R. Punshon, *It Might Lead Anywhere* (1946), p. 148. [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES.

2004. *Evaluation of* π.

In Note 1889 (XXX, May 1946) Mr. D. F. Ferguson gave an account of his new calculation of π, and stated that his figures did not agree with those of Shanks (707 decimals, 1873) after the first 527 decimals.

Mr. Ferguson desired an independent check on his result, and communicated with Professor R. C. Archibald, joint editor of the new American journal, *Mathematical Tables and Aids to Computation.* At Professor Archibald's suggestion, the independent computation was undertaken by Dr. John W. Wrench, assisted by Mr. Levi B. Smith ; and an interesting item in *Mathematical Tables and Aids to Computation,* II, No. 18, April 1947, contains full details and reports by Messrs. Wrench and Smith and by Mr. Ferguson. The new approximations have been carried to 810 decimals, and tally. Readers may be interested to have the agreed figures :

$\pi = 3{\cdot}14159$	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	594(55)								

2005. *On Note* 1942.

My intention in this note was to draw attention to Routh's treatment, which implicitly assumes the convergence of

$$\Sigma\Sigma \frac{(\sin n\pi h/a \sin n'\pi k/b)(\sin n\pi x/a \sin n'\pi y/b)}{\pi^2(n^2/a^2+n'^2/b^2)-q^2}. \qquad \text{(i)}$$

After some juggling he produces a series which, when $h=\frac{1}{2}a$, $k=\frac{1}{2}b$, becomes the divergent series

$$\sum_{n' \text{ odd}} \frac{1}{\phi} \tanh \tfrac{1}{2}\pi\phi, \quad \text{where} \quad \pi^2 n'^2/b^2 - q^2 = \pi^2\phi^2/a^2.$$

Of course, the trouble arises in the first line, since (i) is divergent when $x=h=\frac{1}{2}a$, $y=k=\frac{1}{2}b$; but it is natural to suspect the juggling, and only after careful checking is it realised that the trouble arises from (i). Not wishing to give the show away entirely, I considered the series

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \{(2r+1)^2+(2s+1)^2\}^{-1}, \qquad \text{(ii)}$$

which is analogous to Routh's but pruned of superfluous mathematical frills ; and a similar argument again produces a divergent series,

$$\sum_{n' \text{ odd}} \frac{1}{n'} \tanh \tfrac{1}{2}\pi n',$$

but this time it is fairly obvious that the original double series is divergent, and a clue to Routh's error is exhibited.

The confident assertion in my note that we know that the double series (ii) is absolutely convergent was a deliberate trap, but I hope that my attempt to be entertaining has not wasted too much time on the part of readers looking for the point of the paradox. T. J. WILLMORE.

2006. *A mode of multiplying.*

In each of the examples below the two numbers to be multiplied, which have been chosen at random, appear at the heads of the respective columns. Each succeeding number in the first column of the two examples is obtained by multiplying the number next above it by 2 for example A and by 3 for example B; each succeeding number in the second column of the two examples by dividing the number next above it by 2 for example A, 3 for example B, and taking the result to the nearest integer, numbers containing the fraction one half being taken to the next lower integer. The first method has often been mentioned in the *Gazette*; see, for example, G. H. Bryan (IX, p. 9) on "Russian peasant multiplication".

Then for example A the product of the two numbers is obtained by adding those numbers in the first column which are opposite odd numbers in the second; for example B by subtracting the sum of all numbers in the first column opposite numbers of the form $3n-1$ in the second from the sum of all numbers in the first column opposite numbers of the form $3n+1$ in the second (n an integer).

Example "*A*."		*Example* "*B*."	
Col. 1.	*Col.* 2.	*Col.* 1.	*Col.* 2.
39	35	39	35
78	17	117	12
156	8	351	4
312	4	1053	1
624	2		
1248	1		

$39 \times 35 = 1248 + 78 + 39$
$= 1365$

$39 \times 35 = (1053 + 351) - 39$
$= 1365$

P. C. WICKENS.

2007. *Geodesic-isometric systems on surfaces.*

By a *geodesic-isometric system* I mean a set of isometric curves, one family of which are geodesics, the other family being (necessarily) the geodesic parallels. There would appear to be no limit to the number of different isometric systems that can be drawn on a given surface—at any rate this is true if at least one such system can be found. Systems of geodesics and geodesic parallels appear also to be unlimited in number on any one surface. It is therefore interesting to find that the surfaces upon which geodesic-isometric systems can exist are severely limited; and that, in general, only one such system can be drawn on such a surface.

If a system is isometric the arc-element must be reducible to the form,

$$ds^2 = \lambda(dp^2 + dq^2).$$

If, further, the curves $q = \text{const.}$ are geodesics and $p = \text{const.}$ are geodesic parallels, the arc-element must effectively be the same as

$$ds^2 = dp^2 + D^2 dq^2,$$

showing that λ must be a function of p only. We indicate this by writing

$$ds^2 = dp^2 + P^2\,dq^2,$$

where P is a function of p only.

Now this is the form taken by ds when a surface of revolution is referred to its meridians and parallels; and it follows at once that a geodesic-isometric system can exist only on a surface deformable into a surface of revolution.

If K is the second (Gaussian) curvature of the surface, its value is given by the Gauss characteristic equation which, for the above form of the arc-element, is

$$\frac{1}{P}\frac{d^2P}{dp^2} + K = 0.$$

K must thus be zero, constant, or a function of p only. Consider first, the general, and most interesting, case where K is a function of p only. If the surface were one of revolution (it must, as we have seen, be deformable into one) K would be a function of the latitude only; and it follows that $p =$ const. must be a parallel of latitude. On surfaces of revolution the parallels of latitude and their orthogonal trajectories, the meridian geodesics, form, therefore, the only possible geodesic-isometric system.* Suppose now that some suitable surface carries at least two such systems, and let this surface be deformed into one of revolution. No more than one of the two systems can become the mesh of meridians and parallels; and we therefore have a surface of revolution on which meridians and parallels do not form the only geodesic-isometric system. This we have seen to be impossible, and it would therefore appear that, so long as K is neither zero nor constant, a surface can possess, at most, one geodesic-isometric system.

If K is zero (developable surfaces) or constant and positive (surfaces deformable into spheres), it is hardly necessary to show that the number of geodesic-isometric systems is then unlimited. This result may, however, be formally deduced, and at the same time made to cover the less familiar constant negative case (pseudo-spheres), by referring a surface to a system of concurrent geodesics and geodesic parallels. It is then straightforward to deduce† that such a system, which is clearly not unique, is also isometric in the three cases considered.

The application of the above results to geodesy may be of some interest. An orthomorphic (conformal) representation of a surface may be constructed by ensuring a correspondence between an isometric system on the surface with a Cartesian system in the plane. The Mercator projection, which ensures a correspondence between the meridians and parallels and a Cartesian system, is therefore the only orthomorphic projection of the spheroid in which geodesics can be represented by one family of a Cartesian system. If, as is possible, modern methods enable longitudes to be found with sufficient precision to detect a lack of symmetry of the Earth about its axis, it may be necessary to abandon the use of a surface of revolution as a figure of reference for geodetic operations. If the new figure is not deformable into a surface of revolution we will not be able to adopt a fundamental coordinate system which consists of geodesics and geodesic parallels, and which is, at the same time, isometric. The final choice of a system will therefore depend upon which of the two properties of meridians and parallels it is most advantageous to retain.

E. H. Thompson, Lieut.-Col. R.E.

* This result was given by me in the *Empire Survey Review*, Vol. VIII, No. 62, p. 317.

† Forsyth, *Differential Geometry*, § 211.

REVIEWS.

Analytic Geometry. By FRANCIS D. MURNAGHAN. Pp. viii, 402. \$4.35. 1946. (Prentice-Hall)

This is one of the first books of the Prentice-Hall Mathematics Series to be reviewed in the *Gazette*, and it may not be out of place to begin with a warm welcome. The printing is excellent, the diagrams clear, and the production as a whole seems admirable.

The author is Professor of Applied Mathematics in the Johns Hopkins University, where much of the material has been used. As my impression is that the approach is by no means easy, I ought in fairness to say that Professor Murnaghan has found by experience that " the students enjoy this mode of presentation more than the traditional rather formal method ". The treatment is, indeed, novel and might be difficult to incorporate in this country, but teachers will find many points of interest in the development. The book is certainly worthy of a place on the shelves of a school or college library.

Beginning with geometry on a given straight line, the author establishes, in the notation $v(A \to B)$, the vector defined by the " directed segment " $A \to B$. Vector addition on the line comes quickly, and (surprisingly, I think) direction cosines are introduced, though necessarily restricted to the values ± 1. Division of a directed segment in a given ratio then follows, together with an account of " weighting coördinates ", the latter being shown to be independent of the position of the segment on the line.

In the second chapter, these ideas are extended to plane geometry, up to the following theorem which I quote in illustration—v being written to denote the vector $v(O \to P)$:

" *Every single point P of the plane is such that*

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 , \quad c_1 + c_2 + c_3 = 1,$$

where the values of the three numbers c_1, c_2, c_3 are quite independent of the origin O [but depend on the position of base-points P_1, P_2, P_3]. *We term the three numbers c_1, c_2, c_3, whose sum is unity, the weighting coördinates of the point P with respect to the three points P_1, P_2, P_3.*"

Coordinates in the plane have already been established by means of scales based on given vectors $O \to I$ and $O \to J$. The author now obtains the parametric form :

$$x = at + b, \quad y = ct + d$$

for the coordinates of the points on a line, and the second chapter concludes with a discussion of linearly dependent vectors.

The third chapter is headed : " Products of Plane Vectors ; Equations of Lines ; Area of a Triangle." It begins by establishing direction cosines l, m subject to the relation $l^2 + m^2 = 1$, and the " cosine formula " :

$$\cos \theta = l_1 l_2 + m_1 m_2.$$

Next is defined $v^\star$, the complement of the vector v, in the terms :

" *The complement $v^\star$ of v has the same magnitude as v but its direction is $90°$ ahead of v.*"

Hence, if $v \equiv (x, y)$, then $v^\star \equiv (-y, x)$.

The author then defines the *symmetric product* (scalar product) :

$$(v_1/v_2) = x_1 x_2 + y_1 y_2 = (v_2/v_1),$$

and the *alternating product of* v_2 *by* v_1 :

$$[v_1/v_2] = (v_1^\star/v_2) = x_1y_2 - x_2y_1 = -[v_2/v_1].$$

The equation of a straight line can now be given. If P_1, P are two points of the line such that $v(P_1 \to P) \equiv (x - x_1, y - y_1)$ and if the vector $v_1 \equiv (a, b)$ defines the direction perpendicular to the line, then

$$(v_1/v) = 0.$$

After a discussion of various forms of equation and the sign of the perpendicular from a point to a line, we reach a description of the signed area of a triangle :

" *The signed area of the triangle* OP_1P_2 *is* $\frac{1}{2}[v_1/v_2]$ *where* $v_1 = v(O \to P_1)$ *and* $v_2 = v(O \to P_2)$; *the absolute value of the signed area of the triangle* OP_1P_2 *is the area of the triangle* OP_1P_2 ; *when the signed area is positive,* P_2 *lies to our left as we look along* $O \to P_1$, *and when the signed area is negative,* P_2 *lies on our right as we look along* $O \to P_1$."

Short accounts of pencils of lines and polar coordinates conclude the chapter.

In the fourth chapter, on points and vectors in space, these ideas are extended to three dimensions, with full direction cosines and scalar products. The equation of a plane appears in the analogous form $(v_1/v) = 0$. A discussion of linear dependence establishes that " two space vectors $v_1 = v(a_1, b_1, c_1)$ and $v_2 = v(a_2, b_2, c_2)$ are linearly dependent when (and only when) the vector

$$v(b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1)$$

is the zero vector ". In the general case, this is defined as the *vector product* $(v_1 \times v_2)$. The *alternating product*

$$(v_1v_2v_3) \equiv ((v_1 \times v_2)/v_3)$$

is then explained, together with its use in " orienting " a tetrahedron by analogy with the signed area mentioned above. Applications to spherical trigonometry are given in conclusion.

The fifth chapter contains an account of determinants and matrices, following out of the work already given. Two equations :

$$a_1x + b_1y = c_1, \quad a_2x + b_2y = c_2$$

are combined in the single vector equation

$$xw_1 + yw_2 = w_3.$$

Multiplying each side by $w_1^\star$, $-w_2^\star$ respectively, we have the solutions implied by the equations :

$$y[w_1/w_2] = [w_1/w_3],$$

and

$$x[w_1/w_2] = [w_3/w_2],$$

and indeterminacy and inconsistency are explained. The expression $[w_1/w_2]$ is called the *determinant* of the equations.

Matrices are then described, and the result given that, if

$$A \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad B \equiv (w_1, w_2),$$

where v_1, v_2, w_1, w_2 are space vectors of *three* components each, then :

$$AB = \begin{pmatrix} (v_1/w_1) & (v_1/w_2) \\ (v_2/w_1) & (v_2/w_2) \end{pmatrix}.$$

The solution of linear equations in two and three variables follows, with reference to square matrices, adjoint matrices, reciprocal matrices and rank.

It is shown that, if w_1, w_2, w_3 are given vectors, the matrix $A \equiv (w_1, w_2, w_3)$ has a reciprocal when, and only when, the alternating product $(w_1 w_2 w_3)$ vanishes. This is defined as the *determinant* $D(A)$ of A. The chapter concludes with sections headed "The numerical evaluation of determinants", "Determinants of any number of rows" and "Applications of determinants to problems of analytic geometry".

Having given in some detail the basis on which the book is founded, I pass more quickly over the applications. Subsequent chapter headings are: The Circle and the Sphere; The Parabola; The Ellipse; The Hyperbola; Second Degree Surfaces; The General Equation of the Second Degree in two Variables; The General Equation of the Second Degree in Three Variables. The reader must go to the book itself for details, but a few results on circle and sphere may serve to whet the appetite:

If v, w denote the vectors $v(O \to P)$, $v(O \to C)$, where the length of CP is r, then the equation of the circle with centre C and radius r is

$$(v - w/v - w) = r^2.$$

The equation of the tangent at P_1, where $v(O \to P_1)$ is denoted by v_1, is

$$(v - w/v_1 - w) = r^2.$$

The power of a point P_1 in general position is

$$(v_1 - w/v_1 - w) - r^2.$$

The notation can, however, get awkward (or, at any rate, appear so to one not soaked in it). Thus the inverse of the "general" circle

$$(\boldsymbol{v} - \boldsymbol{w}/\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{R}^2$$

with respect to the circle $$(v/v) = r^2$$

is $$\left(\frac{r^2 v}{(v/v)} - w \Big/ \frac{r^2 v}{(v/v)} - w\right) = R^2,$$

or, in an alternative form,

$$\{(w/w) - R^2\}(v/v) - 2r^2(v/w) + r^4 = 0.$$

Again, the angle θ between two circles of centres c_1, c_2 and radii r_1, r_2 is given by the equation

$$(v(P \to c_1)/v(P \to c_2)) = r_1 r_2 \cos\theta,$$

leading, however, to the more recognisable form

$$2r_1 r_2 \cos\theta = 2(w_1/w_2) - c_1 - c_2.$$

Such, then, is a brief account of an approach of which the author says: "The times are moving so rapidly that we believe the methods used here are not too advanced for college Freshmen, and in fact are suitable for presentation in high school." Without his experience of teaching by this method, I find it hard to believe that a greater maturity of outlook is not required. On the other hand, this is a courageous attempt to eliminate unnecessary detail and to prune away much that the subject can well do without. The argument *marches* across the whole book; and that is no mean feat.

The work is clearly expressed, but I share the author's fear (expressed in the preface) that "in many places the explanations are perhaps rather lengthy". The notation, too, seems confusing at times, though that difficulty might disappear with intensive study. It is, at any rate, certain that geometers would benefit by a study of the methods advocated, even though their own applications of them moved into different channels. E. A. MAXWELL.

Calculus. By L. M. Kells. Pp. vii, 511. $5.00. 1947. (Prentice-Hall)

Applied Industrial Mathematics. By O. B. Jones. Pp. viii, 342. $4.00. 1947. (Prentice-Hall)

These two books, despite some superficial dissimilarities, are basically alike in style and effectiveness. An emphasis on the practical aspects of the work involved, examples of applications chosen within the personal experiences of students likely to use the volumes, and an abundance of illustrations, are to be found in each case. And in each case the methods may be deemed successful.

Professor Kells takes the line that the student of calculations will make applications at the start more easily than at any other time, for he is then "unhampered by complicated mathematical manipulations, and still under the spell of initial curiosity and interest". It is at least arguable that the early technique is as complicated to the beginner as later stages may be to the more advanced learner—and also, probably, that the true mathematician's curiosity and interest are retained indefinitely. Whatever one's views on these points, it is indisputable that an attractive text is here provided.

There is little that is novel in the selection of the contents. Differentiation and integration, applications in geometry and mechanics, multiple integrals, infinite series and ordinary differential equations are the main sections.

The second book is, by British standards, less conventional in its outlook. Mr. Jones' thesis suggests "that a text should not be a storehouse of knowledge but a mental gymnasium, dedicated to the development of mathematical ingenuity". He has written this very general work after many years of experience of the needs of artisans and mechanics in the metal trades, tool engineers and designers, and draughtsmen.

References to practical work we may therefore expect to be plentiful and accurate, and this indeed is so. A considerable variety of information is woven into the pages. Elementary algebra, trigonometry and geometry are mingled together, and with them much work on, for instance, mechanical advantage and horsepower, gears, simple electricity, and so on. The course is painstakingly developed, and the large book covers relatively little ground. The thoroughness of the treatment, however, makes it very suitable for a student working on his own; and it is certain that such a student would gain a real understanding of what he had done.

Reliance on a formula is nowhere advocated; rather is the reader encouraged to think for himself. Worked examples are clearly explained, and there is an adequate supply of exercises.

F. W. K.

Manual of Mathematics and Mechanics. By G. R. Clements and L. T. Wilson. 2nd edition. Pp. ix, 349. 16s. 6d. 1947. (McGraw-Hill)

Those interested in the discussion on Typography that formed part of the Annual Meeting of the Association in January, 1948, will find an added appeal in this book. It contains "certain facts and formulae that are useful in mathematics and mechanics in colleges and engineering schools". The selection is, in general, completely orthodox. Some 90 pages of numerical tables, and 40 pages of tables of integrals, are adequate and accurate. Then follow notes on series, formulae from plane and solid geometry, algebra, trigonometry, calculus and mechanics.

One might note that the compilers suggest that certain topics are outlined more completely than is usual in such manuals; and they instance the solution of plane and spherical triangles, and the solution of ordinary differential equations. It is doubtful, however, if the latter section is, even now, any too full for use in courses in this country.

But it is the setting out of the whole that is so admirable. The preface states that the arrangement and printing are such as to aid rapid work with minimum eye-strain. Hence the reference, above, to the discussion at the Annual Meeting. It is clear that compilers, publisher and printer have collaborated in making the book pleasant to handle, to look at and to use.

It should be added that the last sixty pages of the book contain definitions and notes that will be especially helpful to students of physics. These are of the same high standard as the rest of the volume. F. W. K.

Lehrbuch der darstellenden Geometrie. By EDUARD STIEFEL. Pp. 173. Fr. 28.50. 1947. (Birkhäuser, Basel)

This is a quite excellent textbook on both the theory and the practice of descriptive geometry, a subject which seems to be somewhat neglected in this country.

The book is divided into four parts. The first contains an elementary exposition, covering the straight line and circle, of the method of plan and elevation and of pictorial projection by means of orthogonal axonometry. This is followed by a lengthy and somewhat difficult treatment of constructional problems connected with curved surfaces and curves in space. In the second part, the theory of conics and quadrics is developed, starting from the idea of reciprocation with respect to a circle. Thus conics are defined as reciprocal figures of circles, which at once leads to their equations in polar coordinates. This treatment certainly deserves consideration, but on the whole it seems to necessitate lengthier proofs than the more usual methods of developing the theory. Its advantage lies in the immediate introduction of the principle of duality. Some of the constructions which follow are interesting and not well known.

The third part generalises what has gone before and on the basis of projective geometry builds up the general theory of perspective, treating isometry, military perspective and parallel perspective as special cases. The English reader must be warned here that the cross-ratio which in England is written $(ABCD)$ is in this book written $(ACBD)$. There follows an interesting section on photogrammetry with special reference to its use in aerial photography. The last part deals with spherical descriptive geometry, map projections, and conformal representation, and there is a short appendix on topology.

The book does not assume much previous knowledge beyond a little coordinate geometry, but that does not mean that it is always easy to read. Some passages need close study for thorough comprehension, but they are well worth it. It is probably of more use to the mathematician than to the engineer, for whom insufficient practical details are given and who, in any case, is already well served by several American and English books. On the other hand, no such comprehensive modern treatise on the theory of the subject seems to exist in the English language and, as the subject has developed considerably in the last forty years, such classics as A. E. Church's *Elements of Descriptive Geometry* are now somewhat out-of-date.

The drawings are excellent, though it is unfortunate that several of them are not on the same page as the text belonging to them. There is a definite lack of a glossary, and axioms and theorems are not always kept as distinct as they ought to be. But apart from these minor blemishes the book can be thoroughly recommended. It ought to do much to raise descriptive geometry from the status of " fit only for engineers " to that of " to be studied by mathematicians ". L. R. B. E.

Tables numériques universelles des laboratoires et bureaux d'étude. Opérations arithmétiques, expressions trigonométriques, exponentielles, probabilités, grandeurs réelles et complexes, calcul des formules usuelles, conversion des unités. By Marcel Boll. Pp. iv, 884. $18\frac{1}{2} \times 27$ cm. 3200 fr. 1947. (Dunod, Paris)

This large volume with its many tables covering a very wide field immediately invites comparison with the well-known *Tables of Functions* of Jahnke and Emde (Teubner, Leipzig and Berlin, 1909, 1933, 1938 ; Dover Publications, New York, 1943, 1945) although the ground covered by the two books is not quite the same. Jahnke and Emde give a greater proportion of the higher mathematical functions, while Boll gives compound interest and related tables, and has a wide range of tables of double entry, for example, of a/b, $\sqrt{ab}$, $2ab/(a+b)$, etc.

Boll divides his tables into six groups :

A.	Arithmetic, Algebra	40 sections	Pages 10–218
T.	Trigonometry	38 ,,	,, 220–386
E.	Exponentials	44 ,,	,, 388–536
P.	Probabilities	40 ,,	,, 540–686
C.	Complex Numbers	19 ,,	,, 690–746
U.	Units, Constants	36 ,,	,, 748–854

Each section is usually a table of numerical values, often accompanied by a graph or graphs ; sometimes, particularly in group C, it is a list of formulae. The tables give numbers of figures varying from about 3 or 4 to about 8 or 9, the most usual being perhaps 5 or 6 figures.

Contents. It is not possible to give more than a very rough idea of the full contents of this volume in a review—even to give the list of tables as it appears in the 10-page " Liste des Tableaux " would give only a rough idea of the full scope of the work. Instead, a few fairly typical tables will be described, with a brief indication of other functions that are treated somewhat similarly.

The biggest table in the book is T. 9, pp. 241–285. This gives sin, cos, tan and cot, for every minute of arc from 0° to 45°, mainly to 5 decimals, but with more decimals for small angles. Besides this table, T. 8 gives all six trigonometrical functions to 6 decimals for every degree, T. 10 gives six functions to 5 decimals for 0(0·001)0·25 " turns ", and T. 11 gives four functions to 5 decimals for 0(0·01)5(0·1)10 radians. Nine 5-decimal multiples of the six functions are given for every degree in tables T. 12 to 14, while several functions such as $\sin^2 \alpha$, $\cos^3 \alpha$, $\sqrt{\cos \alpha}$, $(1-\tan \alpha)/(1+\tan \alpha)$, $\sin \alpha \cos^2 \alpha$, $\sin \alpha/\sqrt{\cos \alpha}$, are given for every 30′, with 9 multiples of some of them for every degree.

Other groups of functions fairly fully treated (*i.e.* being given in tables extending over, say, 6 to 15 pages) are powers of integers, logarithms and antilogarithms, natural and hyperbolic, log log and its inverse, factor tables to 10,000, compound interest tables, ascending and descending exponentials, hyperbolic functions, $\cosh x \cos x$, $\cosh x \sin x$, $\sinh x \cos x$, $\sinh x \sin x$, the gudermannian, gamma and related functions, the probability integral, its ordinate and derivatives, Poisson functions.

Smaller tables are given for most other elementary functions that are likely to arise in calculations. Higher functions that are given in smaller tables of two to six pages are the complete and incomplete elliptic integrals, Fresnel's integrals, exponential, sine and cosine integrals, Legendre polynomials, Planck and other radiation functions and integrals, Dalton's integral, some Bessel functions.

Another feature of the volume is the large number of tables of multiples of constants and functions, particularly in Section U. Multiples of sin x and cos x have already been mentioned; there are several others of this kind. Table T. 4 illustrates another type of table; this gives 6- or 7-figure or decimal values of $N\pi/D$, $N/D\pi$, $\pi^{\pm n}$, $(\sqrt{2\pi})^{\pm n}$, $1/2\pi\sqrt{AB}$, $N\sqrt{\pi}/D$, $N/D\sqrt{\pi}$, $N\sqrt{2\pi}/D$, $N/D\sqrt{2\pi}$; each of N, D, n, A, D takes the values 1(1)10. There is also a number of double-entry tables, each giving 3- to 5-figure values and covering about 4 pages. For instance, A. 16 gives b/a, $a=1(1)100$, $b=2(2)30(5)100$; A. 17 gives a/b. A. 18 and A. 19 give the square roots of these quantities. Other functions tabulated are $\sqrt{a^2+b^2}$, $b/\sqrt{a^2+b^2}$, $a/\sqrt{a^2+b^2}$, $\sqrt{ab}$, $2ab/(a+b)$.

Finally a few unusual tables may be noted:

E. 4. Sums of powers of integers, numbers to 100, powers to 7; sums of reciprocals and squares of reciprocals of integers are also given, in A. 9 and A. 10.

E. 12. $x^{\pm x}$, its derivative and integral, $x=0(0{\cdot}01)0{\cdot}05(0{\cdot}05)1{\cdot}5(0{\cdot}1)3(0{\cdot}2)4$.

E. 22. $e^{1/x}$, $e^{-1/x}$, $x^2e^{-1/x}$, $x=0(0{\cdot}2)1(0{\cdot}5)3(1)10(5)30(10)50(25)100$.

E. 23. $e^{-1/w}$ and its derivative,
$w=0(0{\cdot}01)0{\cdot}1(0{\cdot}02)0{\cdot}5(0{\cdot}05)1(0{\cdot}1)5(0{\cdot}5)10(5)50(25)100$.

E. 38. $e^{-u}\sin\alpha$, $\alpha=1^\circ(1^\circ)90^\circ$, $u=0{\cdot}1(0{\cdot}1)1(1)7$.

P. 10. Prime factorials $1 . 2 . 3 . 5 . 7 . 11 \ldots p$ to $p=1009$.

U. 12. $y=\sqrt{e^{2x}-1}$ and $\dfrac{1}{y}$,
$x=0(0{\cdot}001)0{\cdot}017$, $0{\cdot}02(0{\cdot}01)0{\cdot}2(0{\cdot}02)1(0{\cdot}05)2(0{\cdot}1)4$.

Particular mention must be made again of the excellent graphs, often full-page, which accompany many of the tables.

Accuracy. With a table having so wide a scope as this, and having so few rivals in its chosen field much depends on its accuracy. In the case of the tables of Jahnke and Emde, their usefulness is so great and the need for such a collection was so urgent when it appeared, that the existence of errors could be pardoned to some extent, in spite of the fact that there were over 400 of them—known errors were removed from the 1945 edition. Most of the errors in Jahnke and Emde's tables were due to the admitted policy of taking over published tables without checking their accuracy. With Boll's tables, errors of a similar character might also be pardoned, in spite of a much later date of publication, in view of special difficulties during the German occupation of France, when, presumably, the tables were being prepared. No comment whatever seems to have been made about accuracy in the Introduction.

With such considerations in mind, several tables have been examined:

(*a*) As a typical example of a table of a commonly used function, widely tabulated, take the table of log n and ln n (denoted by Log n) to 5 decimals for $n=1(1)1000$; this is part of A. 3, pages 26 to 45. Sixteen errors have been found, mainly major ones.

(*b*) As illustrations of tables taken over from published sources, evidently without checking, consider T. 31, the Fresnel Integrals. This was compared with the table given by Jahnke and Emde. All the errors in the 1933 edition of the latter, including some corrected in the 1938 edition, are present in Boll's table, and a few new discrepancies also occur.

(*c*) As an example of a new table calculated by Boll himself, or under his direction, take A. 25, giving $\sqrt{ab}$. In an examination of half of this table, giving 1450 values, fifteen errors were found. One example will suffice; the

square roots of 48×2, 24×4, 16×6, 12×8, 6×16, are each given as 9·80, while those of 8×12 and 4×24 are *both* given as 9·75.

Another table to note is A. 13, giving smallest divisors of all numbers not divisible by 2, 3 and 5. It is well known that the sequence of units digits 1, 7, 1, 3, 7, 9, 3, 9 recurs regularly ; Boll has omitted 27 arguments (including both primes and composites) in the first half of the table, *i.e.* for $N < 5000$.

As an illustration of an obvious error, see the values of n^4 for $n = 51$ and 55 on page 47. It is difficult to see how this can have been overlooked if any check was applied to this table in proof.

In all, about 7000 values were tested and 90 errors of all kinds located. It is thus an inescapable deduction that the author is totally unaware of the difficulties and pitfalls of computing numerical values and that he has no conception of the steps needed to produce an accurate table—he seems to be ignorant even of such checks as differencing.

The conclusion is that the tables are completely unreliable ; in fact, it seems worthy of comment that no error was found in n^2 and n^3 for $n \leqslant 400$, as far as these were tested!

The diagrams also, in two cases, show small, but obvious discrepancies; for instance, fig. 90, p. 598, in which, at $x = 1{\cdot}5$, $y > 0{\cdot}125$ on the left margin, yet $y < 0{\cdot}125$ on the right margin. A similar discrepancy can be seen in fig. 97, p. 625.

Layout and Typography. In general, the layout of the tables is good, with plenty of space between columns. There are some exceptions, for instance on p. 351, but these are few in number.

The type used is, however, far from good, the numerals being of equal height and uniformly bold for both argument and function. Boll writes " Les caractères de même hauteur sont plus lisibles, par suite de leur parfait alignement : ils ont été pris suffisamment grands et espacés, . . ." This opinion on legibility is not shared by the foremost table makers of the last and present centuries, who favour head and tail figures. In fact, the disadvantages of equal height figures are enhanced by the boldness of the type, as is demonstrated by an analysis of the sixteen errors in the 2000 5-decimal values of log n and ln n. In five of these a 9 replaces a 0 or *vice versa*, in two cases 3 replaces 5 and in two others 6 replaces 8. The numerals of these pairs of digits are very similar in the type used ; no confusion could have arisen with head and tail figures.

Summary. The scope of the work under review is wide, and the plan of the collection is excellent. Although claimed as a completely new conception, the idea of collecting together all tables likely to be needed in numerical work of any kind clearly owes a good deal to the well-known tables of Jahnke and Emde. Diagrams—over one hundred in all—accompany many of the tables, and there is an excellent subject-index, as well as lists of tables and of diagrams.

A one-page Bibliography is given, but consists entirely of French and German works, or of pre-war British and American works. This is understandable in view of the conditions under which it was almost certainly compiled, but with 1947 as publication date, it seems not unreasonable to suggest that it might have been brought rather more up to date.

The layout is, in general, good, although the type used for the numerals seems certain to lead to fatigue and error in use.

In view of the many excellent points about this table, it is unfortunate that the price is so high (though less now, in sterling, than on the publication date), and that it must be condemned as completely unreliable on account of numerous errors.

J. C. P. MILLER.

Algebra for School Certificate and Matriculation. By L. HERMAN and C. ROSS. Part I. Pp. 128. 2s. 6d.; with answers, 2s. 9d. Part II. Pp. 248. 3s. 9d.; with answers, 4s. Part III. Pp. 212. 3s. 6d.; with answers, 3s. 9d. Complete in one volume, 7s. 3d.; with answers, 8s. 1947. (Chambers)

This textbook is written on lines which are now generally reckoned to be out of date. In the first chapter there is a section on the use of letters, and then addition, and substitution are tackled as pure manipulative exercises. There is a mention of a problem in the first paragraph of the book, but after that the text and examples are of the type: add $2b+3c+a$ and similar expressions, or find the value of $\frac{a^{2b}d^{2e}}{b^{2c}}$, when $a=5$, etc. Subsequent chapters are concerned with Brackets, long complicated examples on removal, Directed Numbers, Subtraction, Symbolic Expression, Multiplication, Division, Fractions. It is not till chapters nine and ten that Simple Equations and Problems are introduced. Then there occur in the same chapter examples as different in difficulty as $x+3=6$ and $\frac{x+m}{21+n}=\frac{x-21+m}{n}$. The first part of the book thus goes completely counter to the methods suggested by the Mathematical Association in its report on the Teaching of Algebra. Some applications are used to illustrate the text, but the examples for the pupil only refer to any practical application twice in the first nine chapters, *i.e.* in the chapters on directed number and symbolical expression. Otherwise the examples are pure manipulation, and the child would learn from them "to regard the new subject as meaningless and artificial".*

Part II travels the usual ground from Simultaneous to Quadratic Equations. Factors occur in two chapters. The first gives the methods of grouping, difference of two squares, sum and difference of two cubes. The second is concerned with the factorisation of quadratic expressions. Complicated examples, such as "factorise $9m^2+n^2-6m-2n+6mn+1$" occur in the first chapter. In the second chapter detailed instruction on method is given, using "crosses", the explanation being good for those really interested in the process. But there is no mention of why factorisation is needed, *e.g.* for simplification of complicated Arithmetic expressions, or solving equations.

The third part of the book is concerned with the Theory of Indices, Surds, Logarithms, Progressions, Variation, Theory of Equations. The theoretical bias of the book is further brought out in the chapter on Logarithms, where there is first an explanation of "base", with examples like "Find the value of a if $\log_5 5=a$", before practical examples of multiplication and division with the use of tables. The suggested style of setting out work involves long addition and subtraction across the page, rather than placing one number below another, and this must lead to mistakes or much "side work".

The typography is for the most part excellent. But the questions of worked examples are in the same print as the working, and on page 46, in one example, the working runs on from the question. Examples occur at the end of each chapter, with instructions in the chapter as to when each set is to be attempted. There are a very great number of examples, but no sets of revision papers. There is a full table of contents but no index.

The book covers School Certificate and Matriculation courses, but it is a pity that it is so out of keeping with modern tendencies. It cannot be recommended as a book for beginners, or for normal use in secondary schools.

K. S. S.

* See *The Teaching of Algebra in Schools*, page 12.

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GODFREY HAROLD HARDY
President, 1924–1926